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NUMERICAL IMPLEMENTATION OF A NONLINEAR MODEL OF FLUID FLOW IN A HIGHLY FRACTURED MEDIUM BY THE FINITE ELEMENT METHOD

The paper presents the research results of the iterative method for solving the nonlinear problem of fluid flow in highly porous fractured formations, carried out within the framework of the grant project of the Ministry of Education and Science of the Republic of Kazakhstan No. AP08053189. It is assumed that the fluid flow process in the indicated medium is described by a nonlinear fractional differential equation of anomalous diffusion containing fractional order derivatives in the sense of Caputo-Fabrizio's definition. The first order approximation formula of the fractional derivative is obtained and its properties are investigated. A semi-discrete (with respect to time) and fully discrete finite element schemes of the second order with respect to the spatial variable and the first order with respect to time are constructed. An iterative Newton method is constructed for solving a fully discrete nonlinear equation that occurs at each time step after discretization. The convergence of the method is studied and sufficient conditions for the quadratic convergence of the Newton method are obtained. The results of the theoretical analysis are confirmed by the results of a number of computational experiments.

Keywords*: fluid flow in porous media, finite element method, fractional derivative in the sense of Caputo-Fabrizio, Newton's method, convergence, computational experiments.*

Introduction. For a long time, it was believed that the theory of Muskat-Leverett most fully describes the process of fluid flow in porous media. However, as shown in [1], this theory is not quite adequate in the case of fluid flow in fractured media. On the other hand, it is well known that fluid flow in natural media is usually nonlinear due to the high inhomogeneity of porous media. In such cases, the classical rheological Maxwell, Kelvin-Voigt, or Zener equations do not fully describe the process and require a transition to their fractional differential analogues [2]. In mathematical models of the fluid flow process, fractional order derivatives in the sense of Caputo [1], Riemann-Liouville [2], Caputo-Fabrizio [3], and others were used. The problems of fluid flows, where their dynamics are influenced by a fractured-porous medium and memory effects, are described by the fractional-order integro-differentiation theory in [2, 4]. In [2], several models were proposed to describe fluid flow processes in complex fractured-porous media containing fractional Riemann–Liouville derivatives in time and space. For single-phase flow, a nonlinear pressure equation containing fractional Riemann-Liouville derivatives in time is obtained, a fractional-differential modification of Darcy's law is proposed, and a fractional-differential equation for anisotropic fluid flow is obtained. A fractional-differential modification of the Barenblatt–Gilman model for nonequilibrium two-phase countercurrent capillary impregnation is also proposed, taking into account the effects of power memory when the system relaxes to a local equilibrium state. The classical equations describing the motion of a fluid in a porous medium in [1, 5] were rewritten taking into account the memory formalism using the fractional derivative in the sense of Caputo. The main contribution of [5] is that for the two-phase flow of an incompressible and immiscible fluid in porous media, fractional derivatives of various orders in the Caputo sense with a variable lower limit in fractured and matrix regions were applied, and a two-level discrete time method was also introduced and developed. In [6], a nonlinear two-dimensional orthotropic fluid flow equation with a fractional Riemann–Liouville derivative in time is considered. In [3], the phenomenon of longitudinal dispersion in the flow of two miscible fluids through a porous medium is studied using the Caputo-Fabrizio fractional derivative.

It is known that the derivative considered in [1] has a degenerate singular kernel which makes it difficult to apply approximate methods for its solution. Later in 2015, a new derivative in the sense of Caputo-Fabrizio appeared which is devoid of these shortcomings. The latest features of the Caputo–Fabrizio fractional derivative operator provide more realistic models that help one better adjust the dynamic behavior of real phenomena, as discussed in [1, 3].

Due to the complexity of the application of analytical methods for obtaining solutions to fractional differential equations, the main method for solving such problems, especially nonlinear ones, remains numerical methods. There are many works devoted to the development and study of finite difference methods with various approaches of discretization of fractional derivatives [8, 9]. Other numerical, in particular, finite element [10, 11] and finite volume [12, 13] approaches are also being developed. Fully discrete nonlinear problems are usually solved by various methods. For example, the Newton's iterative method [14, 15], first-order linearization schemes [16], or the Jaeger-Kachur scheme [17].

The aim of this paper is to prove the applicability of the Newton's iterative method for solving a nonlinear differential equation of fluid flow in highly fractured media with a fractional derivative in the sense of Caputo–Fabrizio's definition. We propose a finite element method for implementing a one-dimensional nonlinear fluid flow model in fractured media. Semi-discrete and fully discrete schemes of the second order with respect to the spatial variable and the first order with respect to time are constructed. The Newton's iterative process for solving nonlinear systems of algebraic equations arising when implementing a fully discrete scheme at each time step is constructed. The convergence of the method is analyzed and sufficient conditions for its quadratic convergence are obtained.

Model of fluid flow in a fractured porous medium. In the classical theory of fluid flow in porous media, the continuity equation under the assumption of a single-phase flow of an isothermal fluid in a homogeneous porous medium has the form

$$\partial_t(\phi\rho) + \nabla\cdot(\rho\vec{u}) = f, \qquad (1)$$

where ϕ is the porosity of the medium, ρ is the fluid density, $\vec{u}$ is the velocity vector, f is density of mass sources. One approach to modeling fluid flow in a fractured porous medium is to replace the specified medium with some model homogeneous porous medium with power-law memory. Due to the fact that porosity depends on the pressure of the fluid and on the stress-strain state of the medium, which exhibits viscoelastic properties, it can be concluded that porosity is a function not only of pressure, but also of its fractional derivative or fractional integral [2]:

$$\phi = \phi\left(p, \partial_{0,t}^{\alpha} p\right), \ \alpha \in (-1,1), \tag{2}$$

where the operators of fractional differentiation and integration in the sense of the Caputo-Fabrizio's definition [18, 19] are defined as:

$$\partial_{0,t}^{\nu} u(t) = \frac{M(\nu)}{n-\nu} \int_0^t u^{(n)}(\tau) \exp\left(-\frac{\nu - n + 1}{n - \nu}(t-\tau) \right) d\tau, \quad n-1 < \nu < n, \quad n \in N,$$

$$I_{0,t}^{\nu} u(t) = \frac{2(1-\nu)}{M(\nu)(2-\nu)} u(t) + \nu \int_0^t u(\tau) d\tau, \quad 0 < r < 1, \quad n \in N,$$

where $M = M(r)$ is a function such that $M(0) = M(1) = 0$. In addition, the following fractional-differential generalization of the classical state equation is proposed in [20]:

$$\rho = \rho\left(p, \partial_{0,t}^{\beta} p\right), \quad \beta \in (-1,1). \tag{3}$$

To account for the effect of fractures on the fluid flow process, several generalizations of linear Darcy's law are known. For example, in [1], the motion law is generalized under the assumption that the permeability decreases with time, and therefore the influence of fluid pressure on the flow in a porous medium at the boundary slows down, and the flow occurs as if the medium had a memory. On the other hand, a fractional Darcy law with spatial memory based on the Riemann-Liouville fractional derivative is proposed in [7]. In this paper, the subdiffusion law of motion is chosen as follows:

$$\vec{U} = -\Phi\left(\partial_{0,t}^{\gamma} (\nabla p)\right) \frac{\nabla p}{|\nabla p|}, \quad \gamma \in (0,1), \tag{4}$$

which can be used to describe fluid flow in natural fractured-porous media, in which the fractures are distributed on average evenly over the volume [2], where $\Phi(z)$ is a given function. Substituting (2)-(4) into (1), we obtain the following nonlinear fractional differential equation for the flow of a viscoelastic fluid in a fractured-porous medium under the assumption of a small pressure gradient:

$$\phi\left(c_{f1} + c_{\phi 1}\right) \partial_t p + \phi c_{\phi\alpha} \partial_{0,t}^{\alpha+1} p + \phi c_{f\beta} \partial_{0,t}^{\beta+1} p - \nabla \cdot \left(\frac{\Phi\left(\partial_{0,t}^{\gamma} (\nabla p)\right)}{|\nabla p|} \nabla p \right) = f_0, \tag{5}$$

where $f_0 = \dfrac{f}{\rho}$; $c_{f1} = \dfrac{1}{\rho}\dfrac{\partial\rho}{\partial p}$, $c_{\phi 1} = \dfrac{1}{\phi}\dfrac{\partial\phi}{\partial p}$ are classical isothermal compressibility of a fluid and a porous medium, and $c_{f\beta} = \dfrac{1}{\rho}\dfrac{\partial\rho}{\partial\left(\partial_{0,t}^{\beta} p\right)}$, $c_{\phi\alpha} = \dfrac{1}{\phi}\dfrac{\partial\phi}{\partial\left(\partial_{0,t}^{\alpha} p\right)}$ are their generalized fractional-differential isothermal counterparts [2].

Let us study the special one-dimensional case of the model (1)-(4) in more detail. In $Q_T = \overline{\Omega}\times\left[0,T\right]$, where $\Omega = (0,1)$, consider the initial boundary value problem

$$\partial_t p + \overline{c}_{\phi\alpha}\partial_{0,t}^{\alpha+1} p + \overline{c}_{f\beta}\partial_{0,t}^{\beta+1} p - \overline{k}\left(\Phi\left(\partial_{0,t}^{\gamma}\left(p_x\right)\right)\right)_x = \overline{f}_0, \quad t > 0, \quad x \in \Omega, \tag{6}$$

$$p\left(x,0\right) = p_0\left(x\right), \quad x \in \overline{\Omega}, \tag{7}$$

$$p\left(0,t\right) = p\left(1,t\right) = 0, \ t > 0. \tag{8}$$

First, we define a variational formulation of the problem (6)-(8).

Problem 1. Find $p \in H^1\left(0,T;H_0^1\left(\Omega\right)\right)$, such that for any $v \in H_0^1\left(\Omega\right)$:

$$\left(\partial_t p, v\right) + \overline{c}_{\phi\alpha}\left(\partial_{0,t}^{\alpha+1} p, v\right) + \overline{c}_{f\beta}\left(\partial_{0,t}^{\beta+1} p, v\right) + \overline{k}\left(\Phi\left(\partial_{0,t}^{\gamma}\left(p_x\right)\right), v_x\right) = \left(\overline{f}_0, v\right), \tag{9}$$

$$p\left(x,0\right) = p_0\left(x\right), \tag{10}$$

where $\alpha \in (-1,0)$, $\beta \in (-1,0)$, $\gamma \in (1,0)$.

Throughout the paper, we use the following assumptions:

(A1) The problem (6)-(8) has a unique solution that has the number of derivatives required to perform the analysis.

(A2) The function Φ: $R \to R$ is a differentiable real-valued function defined on Ω, such that

$$\Phi'\left(z\right) \geq c_f > 0, \quad z \in R, \tag{11}$$

$$\left|\Phi\left(z_1\right) - \Phi\left(z_2\right)\right| \leq L_f\left|z_1 - z_2\right|, \quad L_f > 0, \quad z_1, z_2 \in R. \tag{12}$$

(A3) The initial condition p^0 is bounded and positive.

Semi-discrete and fully discrete formulations of the problem. Introduce a partition of the time interval $[0, T]$ by the points $t_n = n\tau$, $n = 0,1, ..., N_t$, $N_t\tau = T$. To determine the semi-discrete formulation of (6), (7), we derive the approximation formula of the fractional derivative in the sense of Caputo-Fabrizio's definition.

Lemma 1. The discrete analog of the Caputo-Fabrizio fractional derivative of order ν can be represented as:

$$\partial_{0,t}^{\nu} p\left(t_n\right) = \Delta_{0,t}^{\nu} p\left(t_n\right) + r_n^{\nu}, \quad 0 < \nu < 1, \tag{13}$$

$$\Delta_{0,t}^{\nu} p(t_n) = \sum_{s=1}^{n} \delta_{n,s}^{\nu} \left(p^s - p^{s-1}\right), \tag{14}$$

where p^s denotes a finite-dimensional approximation of the function p at point $t = t_s$,

$$\delta_{n,s}^{\nu} = \frac{\exp(\sigma_\nu \tau) - 1}{\tau\nu} \exp\left(-(n-s+1)\sigma_\nu \tau\right),\ \sigma_\nu = \frac{\nu}{1-\nu},\ \left|r_n^{\nu}\right| \le \frac{\exp(-\sigma_\nu t_n)}{3\nu} \max_{0\le t\le t_n} \left|p''(t)\right| \tau.$$

Define a semi-discrete formulation of the problem (6)-(7):

Problem 2. Let the value of $p^{n-1} \in H_0^1(\Omega)$ be known. Find $p^n \in H_0^1(\Omega)$ such that for all $v \in H_0^1(\Omega)$:

$$\left(\frac{p^n - p^{n-1}}{\tau}, v\right) + \bar{c}_{\phi\alpha}\left(\Delta_{0,t}^{\alpha+1} p^n, v\right) + \bar{c}_{f\beta}\left(\Delta_{0,t}^{\beta+1} p^n, v\right) + \left(\Phi\left(\Delta_{0,t}^{\gamma} p_x^n\right), v_x\right) = \left(\bar{f}_0, v\right), \tag{15}$$

$$p^0 = p_0(x), \tag{16}$$

where $\alpha \in (-1,0)$, $\beta \in (-1,0)$, $\gamma \in (1,0)$.

To formulate a fully discrete problem, we define a discrete space $V_h \subset H_0^1$:

$$V_h = \left\{ v_h \in H_0^1(\Omega) \cap C^0\left(\bar{\Omega}\right) \mid \ v_h|_e \in P_1(e), \quad \forall e \in K_h \right\},$$

where K_h is the quasi-uniform triangulation of Ω. Let Π_h be the L^2 – projection operator, such that $\left(\Pi_h p - p, p_h\right) = 0$, $\forall p \in L^2(\Omega)$, $p_h \in W_h$.

Problem 3. Find $p_h^n \in V_h$, $n = 1,2, ..., N_t$, satisfying

$$\left(\frac{p_h^n - p_h^{n-1}}{\tau}, v_h\right) + \bar{c}_{\phi\alpha}\left(\Delta_{0,t}^{\alpha+1} p_h^n, v_h\right) + \bar{c}_{f\beta}\left(\Delta_{0,t}^{\beta+1} p_h^n, v_h\right) + \left(\Phi\left(\Delta_{0,t}^{\gamma} p_{h,x}^n\right), v_{h,x}\right) = \left(\bar{f}_0, v_h\right), \tag{17}$$

$$p_h^0 = \Pi_h p_0, \tag{18}$$

for any $v_h \in V_h$ where $\alpha \in (-1,0)$, $\beta \in (-1,0)$, $\gamma \in (1,0)$.

Previously, the authors proved the following theorems in the special case $\Phi(\varphi) = \mu\varphi(x,t) + \psi(x,t)$.

Theorem 1. Under the conditions (A1)-(A3), the following inequality holds:

$$\left\| p(t_n) - p^n \right\|_0 + \tau\sqrt{\frac{2c_0}{T}} \left\| p(t_n) - p^n \right\|_1 \le C\tau,$$

where $c_0 = \min\left\{ \bar{c}_{\phi\alpha}\delta_{n,1}^{\alpha+1}, \bar{c}_{f\beta}\delta_{n,1}^{\beta+1}, \mu\delta_{n,1}^{\gamma} \right\}$.

Theorem 2. Let $p \in C^1\left([0,T], H_0^1(\Omega) \cap H^2(\Omega)\right)$, $p_h^n \in V_h$ and $p_h^0 = \Pi_h p_0$. Then there exists $\tau_0 > 0$ such that for all $\tau \le \tau_0$, the following inequality holds:

$$\left\| p(t_n) - p_h^n \right\|_0 + 2\tau\sqrt{\frac{c_0}{T}} \left\| p(t_n) - p_h^n \right\|_1 \le C\left(\tau + h^2\right),$$

where $c_0 = \min\left\{ \bar{c}_{\phi\alpha}\delta_{n,1}^{\alpha+1}, \bar{c}_{f\beta}\delta_{n,1}^{\beta+1}, \mu\delta_{n,1}^{\gamma} \right\}$.

For the more general case of Φ considered in the paper, the method converges only with the first order in the spatial variable with an additional restriction on the diameter of the triangulation.

Implementation of the Newton's method. In this section, we present a Newton's method for solving the nonlinear problem (6)-(8). Rewrite (15), using (14):

$$C_{n,\tau}^{\alpha,\beta}\left(p^n - p^{n-1}, v\right) + \tau\bar{k}\left(\Phi\left(\delta_{n,n}^{\gamma} p_x^n + H_n^{\alpha,\beta}\right), v_x\right) = \left(F_n, v\right), \tag{19}$$

where

$$C_{n,\tau}^{\alpha,\beta} = 1 + \tau\delta_{n,n}^{\alpha+1}\bar{c}_{\phi\alpha} + \tau\Gamma_{n,n}^{I+1}\bar{c}_{f\beta},\ H_n^{\alpha,\beta} = -\delta_{n,n}^{\gamma} p_x^{n-1} + \sum_{s=1}^{n-1}\delta_{n,s}^{\gamma}\left(p_x^s - p_x^{s-1}\right),$$

$$F_n = -\tau\bar{c}_{\phi\alpha}\sum_{s=1}^{n-1}\delta_{n,s}^{\alpha+1}\left(p^s - p^{s-1}\right) - \tau\bar{c}_{f\beta}\sum_{s=1}^{n-1}\delta_{n,s}^{\beta+1}\left(p^s - p^{s-1}\right) + \tau\bar{f}_0.$$

Denote the i-th iteration by $p^{n,i}$ and, limiting ourselves to the first two terms in the Taylor expansion of Φ, we obtain

$$C_{n,\tau}^{\alpha,\beta}\left(p^{n,i} - p^{n-1}, v\right) +$$

$$+\tau\bar{k}\left(\Phi\left(\delta_{n,n}^{\gamma} p_x^{n,i-1} + H_n^{\alpha,\beta}\right) + \delta_{n,n}^{\gamma}\left(p_x^{n,i} - p_x^{n,i-1}\right)\Phi'\left(\delta_{n,n}^{\gamma} p_x^{n,i-1} + H_n^{\alpha,\beta}\right), v_x\right) = \left(F_n, v\right). \tag{20}$$

Lemma 2 [14]. Let $f : R \to R$ be a differentiable function, and $f'(\cdot)$ be Lipschitz continuous. Then

$$\left\| f(x) - f(y) - f'(y)(x-y)\right\|_0 \le \frac{L_{f'}}{2}\left\|x-y\right\|_0^2, \qquad \forall x, y \in R,$$

where $L_{f'}$ is the Lipschitz constant.

Theorem 3. Under the conditions (A1)-(A3), the following inequality holds:

$$\left\|p^n - p^{n,i}\right\|_0^2 + \tau\left\|p_x^n - p_x^{n,i}\right\|_0^2 \le \frac{C\tau}{h^5}\left\|p^n - p^{n,i-1}\right\|_0^4.$$

Proof. Subtracting (20) from (19), we obtain the following identity:

$$C_{n,\tau}^{\alpha,\beta}\left(p^n - p^{n,i}, v\right) + \tau\bar{k}\left(\Phi\left(\delta_{n,n}^{\gamma} p_x^n + H_n^{\alpha,\beta}\right) - \Phi\left(\delta_{n,n}^{\gamma} p_x^{n,i-1} + H_n^{\alpha,\beta}\right) -\right.$$

$$\left.-\delta_{n,n}^{\gamma}\left(p_x^n - p_x^{n,i-1}\right)\Phi'\left(\delta_{n,n}^{\gamma} p_x^{n,i-1} + H_n^{\alpha,\beta}\right), v_x\right) + \tau\bar{k}\left(\delta_{n,n}^{\gamma}\left(p_x^n - p_x^{n,i}\right)\Phi'\left(\delta_{n,n}^{\gamma} p_x^{n,i-1} + H_n^{\alpha,\beta}\right), v_x\right) = 0\ .$$

Denote $e^{n,i} = p^n - p^{n,i}$ and choose $v = e^{n,i}$:

$$C_{n,\tau}^{\alpha,\beta}\left(e^{n,i}, e^{n,i}\right) + \tau\bar{k}\left(\Phi\left(\delta_{n,n}^{\gamma} p_x^n + H_n^{\alpha,\beta}\right) - \Phi\left(\delta_{n,n}^{\gamma} p_x^{n,i-1} + H_n^{\alpha,\beta}\right) -\right.$$

$$\left.-\delta_{n,n}^{\gamma} e_x^{n,i-1}\Phi'\left(\delta_{n,n}^{\gamma} p_x^{n,i-1} + H_n^{\alpha,\beta}\right), e_x^{n,i}\right) + \tau\bar{k}\left(\delta_{n,n}^{\gamma} e_x^{n,i}\Phi'\left(\delta_{n,n}^{\gamma} p_x^{n,i-1} + H_n^{\alpha,\beta}\right), e_x^{n,i}\right) = 0\ \cdot$$

Using assumption (A2) and Lemma 2, we obtain

$$C_{n,\tau}^{\alpha,\beta}\left\|e^{n,i}\right\|_0^2+\tau\bar{k}c_0\delta_{n,n}^{\gamma}\left\|e_x^{n,i}\right\|_0^2\le\tau\delta_{n,n}^{\gamma}\bar{k}\int_{\Omega}\frac{L_{f'}}{2}\left|e_x^{n,i-1}\right|^2\left|e_x^{n,i}\right|dx\le$$

$$\le\frac{\tau}{8c_0}\delta_{n,n}^{\gamma}\bar{k}\left(L_{f'}\right)^2\left\|e_x^{n,i-1}\right\|_{L^4(\Omega)}^4+\frac{c_0}{2}\tau\delta_{n,n}^{\gamma}\bar{k}\left\|e_x^{n,i}\right\|_0^2,$$

which yields

$$C_{n,\tau}^{\alpha,\beta}\left\|e^{n,i}\right\|_0^2+\frac{c_0\tau}{2}\bar{k}\delta_{n,n}^{\gamma}\left\|e_x^{n,i}\right\|_0^2\le\frac{\tau}{8c_0}\delta_{n,n}^{\gamma}\bar{k}\left(L_{f'}\right)^2\left\|e_x^{n,i-1}\right\|_{L^4(\Omega)}^4.$$

Using the inverse inequalities $\|v\|_{L^4(\Omega)}\le Ch^{-\frac{d}{4}}\|v\|_{L^2(\Omega)}$ and $\|v_x\|_0\le Ch^{-1}\|v\|_0$, we obtain

$$C_{n,\tau}^{\alpha,\beta}\left\|e^{n,i}\right\|_0^2+\frac{c_0\tau}{2}\bar{k}\delta_{n,n}^{\gamma}\left\|e_x^{n,i}\right\|_0^2\le\frac{C\tau}{h}\left\|e_x^{n,i-1}\right\|_0^4$$

which concludes the theorem.

It follows from the theorem that if a sufficient condition is satisfied $\frac{C\tau}{h^5}\left\|p^n-p^{n,i-1}\right\|_0^2\le1$ the quadratic convergence of the Newton's method is achieved.

Computational experiments. To check the accuracy of the scheme (20), a number of computational experiments were carried out on the example of two model problems.

Example 1. Consider the equation

$$\partial_t p+\partial_{0,t}^{\alpha+1}p+\partial_{0,t}^{\beta+1}p-\left(\arctan\left(\partial_{0,t}^{\gamma}\left(p_x\right)\right)\right)_x=f_0,\ t>0, \tag{21}$$

$$f_0(x,t)=x(x-1)\left[\frac{\exp((\alpha+1)t/\alpha)-\exp(t/2)}{\alpha+2}+\frac{\exp((\beta+1)t/\beta)-\exp(t/2)}{\beta+2}-\frac{\exp(t/2)}{2}\right]-$$

$$-\frac{2(\gamma+1)\left(\exp(t\gamma/(\gamma-1))-\exp(t/2)\right)}{\left(\exp(t)-2\exp(t\gamma/(\gamma-1)+t/2)+\exp(2t\gamma/(\gamma-1))\right)\left(4x^2-4x+1\right)+\gamma^2+2\gamma+1}$$

with the initial and boundary conditions

$$p(x,0)=x(1-x),\ x\in\bar{\Omega},\ \ p(0,t)=p(1,t)=0,\ t>0, \tag{22}$$

where $\alpha\in(-1,0)$, $\beta\in(-1,0)$, $\gamma\in(1,0)$. The exact solution of the problem is $p(x,t)=$ $=x(1-x)\exp(t/2)$.

Example 2. Consider the equation

$$\partial_t p+\partial_{0,t}^{\alpha+1}p+\partial_{0,t}^{\beta+1}p-\left(\arctan\left(\partial_{0,t}^{\gamma}\left(p_x\right)\right)\right)_x=f_0,\quad t>0, \tag{23}$$

where $f_0(x,t)=\sin(\pi x)\exp(2-t)\left[\frac{\exp((\alpha+1)t/\alpha+t)-1}{2\alpha+1}+\frac{\exp((\beta+1)t/\beta)-1}{2\beta+1}-1\right]+$

$$+\frac{\pi^2(2\gamma-1)\sin(\pi x)\exp(2+t)\left(\exp(t\gamma/(\gamma-1)+t)-1\right)}{\pi^2\cos(\pi x)^2\left(\exp(2t\gamma/(\gamma-1)+2t+4)-2\exp(t\gamma/(\gamma-1)+t+4)+\exp(4)\right)+\exp(2-t)\left(4\gamma^2-4\gamma+1\right)}$$

with the initial and boundary conditions

$$p(x,0)=\sin(\pi x),\ x\in\overline{\Omega},\ \ p(0,t)=p(1,t)=0,\ t>0, \tag{24}$$

where $\alpha \in (-1,0)$, $\beta \in (-1,0)$, $\gamma \in (1,0)$. The exact solution of the problem is $p(x,t)=\sin(\pi x)\exp(2-t)$.

In tables 1 and 2 errors values for different values of the parameters $\alpha=-0.5$, $\beta=-0.5$, $\gamma=0.5$ are shown.

Table 1 – Convergence of the Newton method for the Example 1, with $\alpha=-0.5$, $\beta=-0.5$, $\gamma=0.5$, $h=0.05$, $\tau=0.025$.

$t=0.025$	$t=0.05$	$t=0.125$	$t=0.5$	$t=1$
$2.5824682\cdot10^{-1}$	$1.6148931\cdot10^{-1}$	$3.4512562\cdot10^{-2}$	$1.3654528\cdot10^{-3}$	$9.0365485\cdot10^{-4}$
$2.5559525\cdot10^{-3}$	$1.3654821\cdot10^{-3}$	$2.9604762\cdot10^{-4}$	$6.6548218\cdot10^{-6}$	$9.6542582\cdot10^{-6}$
$1.5852454\cdot10^{-7}$	$9.1236548\cdot10^{-8}$	$9.6548924\cdot10^{-9}$	$5.5852454\cdot10^{-11}$	$6.9655420\cdot10^{-10}$
$7.2354156\cdot10^{-11}$	-	-	-	-

Table 2 – Convergence of the Newton method for the Example 2, with $\alpha=-0.5$, $\beta=-0.5$, $\gamma=0.5$, $h=0.05$, $\tau=0.025$.

$t=0.025$	$t=0.05$	$t=0.125$	$t=0.5$	$t=1$
$2.6352452\cdot10^{-1}$	$1.6525215\cdot10^{-1}$	$2.3256012\cdot10^{-2}$	$1.22565215\cdot10^{-3}$	$1.3295214\cdot10^{-3}$
$1.96532541\cdot10^{-3}$	$1.6521562\cdot10^{-3}$	$1.0215921\cdot10^{-4}$	$1.2159852\cdot10^{-5}$	$1.5625921\cdot10^{-5}$
$5.3562548\cdot10^{-8}$	$4.3832568\cdot10^{-8}$	$4.3255489\cdot10^{-9}$	$9.3265802\cdot10^{-10}$	$5.6254862\cdot10^{-10}$

Conclusions. Thus, a finite element scheme is constructed for the nonlinear fractional-differential equation of anomalous diffusion, which describes the fluid flow process in highly porous fractured formations, containing a fractional derivative in the sense of Caputo-Fabrizio. An approximation formula of the first order of the fractional derivative is obtained and its properties are investigated. The order of convergence depends only on the sampling parameters. At each time step, the resulting nonlinear algebraic systems are solved by Newton's method. In addition, a sufficient condition for the quadratic convergence of the Newton scheme is obtained. To confirm the theoretical results, two numerical examples were presented. The empirical convergence agrees well with the theoretical estimates. Based on the theoretical results and numerical experiments, we conclude that the presented numerical scheme is effective and can be used for effective modeling of fluid flow problems in highly porous fractured formations.

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ЖОҒАРЫ ЖАРЫҚШАЛЫ ОРТАДА СЫЗЫҚТЫ ЕМЕС ФИЛЬТРАЦИЯ МОДЕЛІН АҚЫРЛЫ ЭЛЕМЕНТТЕР ӘДІСІМЕН САНДЫҚ ЖҮЗЕГЕ АСЫРУ

Мақалада ҚР БжҒМ гранттық қаржыландырылған АР08053189 жобасы аясында жүргізілген кеуектілігі жоғары жарықшалы қабаттардағы сызықты емес фильтрация есебін шешудің итерациялық әдісін зерттеу нәтижелері ұсынылған. Жұмыста көрсетілген ортадағы фильтрация үрдісі Капуто-Фабрицио мағынасындағы бөлшек ретті туындылары бар аномальды диффузияның сызықты емес бөлшек-дифференциалдық теңдеуімен сипатталады деп болжанады. Бөлшек туындының бірінші ретті жуықтау формуласы алынды және оның қасиеттері зерттелді. Уақытқа қатысты жартылай дискретті және кеңістіктік айнымалы бойынша екінші ретті және уақыт бойынша бірінші ретті толығымен дискретті ақырлы-элементтік сұлба құрылды. Дискреттеуден кейін әр уақыт қадамында пайда болатын толық дискретті сызықты емес теңдеуді шешу үшін Ньютонның итерациялық әдісі құрылды. Жинақтылық мәселесі зерттелді және Ньютон әдісінің квадраттық жинақтылығының жеткілікті шарттары келтірілді. Теориялық талдау нәтижелері бірқатар есептеу эксперименттерінің нәтижелерімен расталады.

***Түйін сөздер**: кеуекті ортадағы сұйықтықтың ағыны, ақырлы элементтер әдісі, Капуто-Фабрицио мағынасындағы бөлшек туынды, Ньютон әдісі, жинақтылық, есептеу эксперименттері.*

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ЧИСЛЕННАЯ РЕАЛИЗАЦИЯ НЕЛИНЕЙНОЙ МОДЕЛИ ФИЛЬТРАЦИИ В СИЛЬНО ТРЕЩИНОВАТОЙ СРЕДЕ МЕТОДОМ КОНЕЧНЫХ ЭЛЕМЕНТОВ

Представлены результаты исследования итерационного метода решения нелинейной задачи фильтрации в сильнопористых трещиноватых пластах, проводимого в рамках грантового проекта МОН РК АР08053189. В работе предполагается, что процесс фильтрации в указанной среде описывается нелинейным дробно-дифференциальным уравнением аномальной диффузии, содержащим производные дробного порядка в смысле Капуто-Фабрицио. Получена аппроксимационная формула первого порядка дробной производной и исследованы ее свойства. Построены полудискретная относительно времени и полностью дискретная конечно-элементные схемы второго порядка по пространственной переменной и первого порядка по времени. Построен итерационный метод Ньютона для решения полностью дискретного нелинейного уравнения, возникающего на каждом временном шаге после дискретизации. Исследован вопрос о сходимости и приведены достаточные условия квадратичной сходимости метода Ньютона. Результаты теоретического анализа подтверждаются результатами ряда вычислительных экспериментов.

***Ключевые слова**: течение жидкости в пористых средах, метод конечных элементов, дробная производная в смысле Капуто-Фабрицио, метод Ньютона, сходимость, вычислительные эксперименты.*