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STABILIZED FINITE ELEMENT METHOD FOR THE SATURATION EQUATION IN THE TWO-PHASE NONEQUILIBRIUM FLUID FLOW PROBLEM

In this paper, an approximate method for solving the saturation equation in the problem of two-phase nonequilibrium flow in porous media is constructed. This problem is being studied as part of the work carried out under a grant project funded by the Ministry of Education and Science of the Republic of Kazakhstan, Grant No. AP08053189. This equation refers to an equation of the convection-diffusion type with a predominance of convection and with an additional term containing the third-order derivative of the solution. Due to the hyperbolic nature of the equation, its solution is accompanied by a number of difficulties that lead to the need for a careful choice of the solution method. One of the difficulties is the appearance of non-physical oscillations at the interface of the two phases. Three classical stabilized finite element methods (SUPG, GLS and USFEM) are compared based on computational experiments. In addition, comparative calculations were performed using several stabilization parameters due to the sensitivity of the stabilized methods to the choice of these parameters and the significant dependence of stability and accuracy on them. Methodological calculations are carried out and the results of calculations with different values of mesh configurations, stabilization parameters are presented.

***Key words**: finite element method, stabilized method, nonequilibrium flow in porous media, SUPG, GLS, USFEM.*

Introduction. The dynamics of the fluid flows of a multiphase fluid depends in a nonlinear way on both the structural and mechanical properties of the fluid and the properties of the surrounding skeleton. However, in real reservoir conditions, the property of delayed phase saturation has a significant effect on the flow process, the study of which led to the emergence of the theory of non-equilibrium flows. The influence of nonequilibrium can be significant: the time for establishing saturation in the conditions of oil fields is on the order of a year.

One of the main models of nonequilibrium flow [1] is based on thermodynamic arguments and volume averaging of microscopic equations of conservation of mass and moment, which led to the need to add additional terms to the macroscopic equations. In [1], the concept of dynamic capillary pressure P_c^{dyn} (instantaneous local difference between phase pressures) was introduced, which relates to the static capillary pressure P_c^{stat} (capillary pressure under quasi-static displacement) by the ratio

$$P_c^{\mathrm{dyn}} \equiv p_o - p_c = P_c^{\mathrm{stat}} - \tau_H(s)\partial_t s, \tag{1}$$

where p_0 and p_w are the phase pressures of oil and water, τ_{H} is the phenomenological coefficient taking positive values, and s is the water saturation. Dynamic capillary pressure has been the subject of many experimental [2] and theoretical [3, 4] studies.

Taking into account the nonequilibrium law (1), the two-phase nonequilibrium flow problem is reduced to solving a system of partial differential equations for determining pressure and saturation fields. In the paper of the authors [5], an iterative method for solving the pressure equation based on the mixed finite element method is constructed.

The aim of this paper is to develop a finite element method for solving the saturation equation in a two-phase nonequilibrium flow model with the nonequilibrium law proposed in [1]. This equation refers to an equation of the convection-diffusion type with a predominance of convection and with an additional term containing the third-order derivative of the solution. Due to the hyperbolic nature of the equation, its solution is accompanied by a number of difficulties that lead to the need for a careful choice of the solution method. One of them is associated with a jump of the solution at the interface, when the saturation is accompanied by a sharp change in the transition from one zone to another.

It is known [6-8] that the application of the classical Galerkin method to calculate saturation field using the widely used IMPES method in the neighborhood of the gap leads to non-physical oscillations. One way to overcome such oscillations is the streamline upwind Petrov-Galerkin finite element method (SUPG), which was proposed by Brooks and Hughes and later earned the attention of many researchers. The essence of the method is to add an additional artificial viscosity with a certain stabilizing parameter. Currently, quite a few varieties and implementations of the method have been developed. Applications of the SUPG method to the solution of saturation equations are known [9].

Other popular method for solving convection-dominated equations are the Galerkin least squares method (GLS). An essential feature of the GLS method is the modification of the weak form construction for the Galerkin method and acts as a means of stabilizing the fluid flow equations. The GLS is closely related to SUPG, but is a conceptually simpler and more general methodology applicable to a wide range of problem classes. There are known applications of the method to the implementation of the ice cover model [10], incompressible Navier-Stokes equations [11], Maxwell model with upper convection [12] and others.

Another stabilization method leads to the unusual stabilized finite element method (USFEM), proposed in [13]. The main idea of the method is to extend the space of piecewise continuous polynomials by functions defined element-by-element, in such a way as to improve accuracy and stability. The solution is sought as the sum of two solutions from two spaces – the space of linear polynomials and the space of one basis functions called bubble functions. This method is used to solve the stationary convection-reaction problem [14], to implement the large vortex model [15], and many others.

The disadvantages of the stabilized methods include their sensitivity to the choice of stabilization parameters, which significantly affects the stability and accuracy of the method. Therefore, the problem of choosing a parameter should be thoroughly investigated. There are a number of papers [16-18] devoted to the study and comparison of the stabilizing parameters for the Navier-Stokes equations, the convection-diffusion-reaction equation, and others.

This paper compares three classical stabilized finite element methods (SUPG, GLS, and USFEM) to the initial boundary problem for saturation equation, as well as the stabilization parameters based on computational experiments.

Formulation of the problem. In a bounded domain $Q_T = \Omega \times (0,T)$, where $\Omega \subset R^2$ with the boundary $\Gamma = \Gamma_D \cup \Gamma_N$, $\Gamma_D \cap \Gamma_N = \varnothing$, the following initial boundary value problem is considered [1]:

$$\phi \partial_t s + \nabla \cdot \vec{u}_w = q_w, \tag{2}$$

$$-\phi \partial_t s + \nabla \cdot \vec{u}_o = q_o, \tag{3}$$

$$\vec{u}_\alpha = -k k_\alpha(s) \mu_\alpha^{-1} \nabla p_\alpha,\ \alpha \in \{w,o\}, \tag{4}$$

$$p_o - p_w = p_c(s) - L \partial_t s, \tag{5}$$

$$s(x,0) = s_0(x),\ x \in \Omega, \tag{6}$$

$$p(x,t) = p_{\text{inj}},\ (x,t) \in \Gamma_D \times (0,T], \tag{7}$$

$$\nabla p \cdot \vec{n} = 0,\ (x,t) \in \Gamma_N \times (0,T], \tag{8}$$

where $\vec{u}_\alpha = (u_{\alpha 1}(x,t), u_{\alpha 2}(x,t))$ and $p_\alpha = p_\alpha(x,t)$ are the velocity and pressure of the phase α, respectively, $s = s(x,t)$ is the water saturation, $p_c = p_c(s)$ is the capillary pressure, $\vec{n}$ is the external unit normal to the boundary Γ, $L = L(s) \geq 0$.

To derive the computational model, introduce the total velocity vector as follows:

$$\vec{u} = \vec{u}_w + \vec{u}_o. \tag{9}$$

Using (4), we obtain

$$\vec{u} = -k(\lambda_w \nabla p_w + \lambda_o \nabla p_o), \tag{10}$$

where $\lambda_\alpha = k_\alpha \mu_\alpha^{-1}$ is the mobility of the phase α. Introduce a new variable, global pressure p, such that

$$\lambda_w \nabla p_w + \lambda_o \nabla p_o = \lambda \nabla p, \tag{11}$$

where $\lambda = \lambda_w + \lambda_o$ is the total mobility. Using the equations (2)-(5) it is not difficult to write out the explicit form of the variable p:

$$p = h_w p_w + h_o p_o + \frac{1}{2}(h_w - h_o) p_c - \frac{1}{2}\int_{s_c}^{s} (f_w - f_o) p_c'(\xi) d\xi, \tag{12}$$

where $h_w = h_w(s)$ and $h_o = h_o(s)$ are some functions such that $h_w + h_o = 1$, and $f_\alpha = \frac{\lambda_\alpha}{\lambda}$.

To obtain the pressure equation, sum the equations (2) and (3) and use the equations (10) and (11):

$$\begin{gathered} \nabla \cdot \vec{u} = 0, \\ (k\lambda)^{-1} \vec{u} + \nabla p = 0. \end{gathered} \tag{13}$$

It is not difficult to show that the phase velocities are expressed in terms of the total velocity by the relation

$$\vec{u}_w = f_w(s)\vec{u} - \gamma(s)\nabla s - \gamma_1(s)\nabla(L\partial_t s), \tag{14}$$

where $\gamma(s) = -K\lambda_o(s) f_w(s)\frac{dp_c}{ds} > 0$, $\gamma_1(s) = K\lambda_o(s) f_w(s) > 0$. Substituting (14) into (2), we obtain the equation for saturation:

$$\phi\partial_t s + f_w^{'}(s)\vec{u}\cdot\nabla s - \nabla\cdot(\gamma(s)\nabla s) - \nabla\cdot(\gamma_1(s)\nabla(L\partial_t s)) = 0. \tag{15}$$

Thus, a computational model consisting of the equations (13), (15) and the corresponding initial and boundary conditions is obtained.

In [5], a finite element method for solving equations (13) is constructed, and the convergence of the method is investigated and its a posteriori analysis is carried out. Let us focus on solving the equation (15) in more detail, assuming the vector $\vec{u}$ is known. Namely, in the domain *QT* defined above, consider the equation

$$\partial_t s + As + B\partial_t s = f,\ (x,t)\in Q_T \tag{16}$$

$$s(x,0) = s_0,\ x\in\overline{\Omega}, \tag{17}$$

$$s = g_D,\ x\in\Gamma_D;\ \nabla s\cdot\vec{n} = g_N,\ x\in\Gamma_N, t>0, \tag{18}$$

where

$$As = \vec{u}\cdot\nabla s - k_1\nabla^2 s,\ Bs = -k_2\nabla^2 s,$$

f, g_D, g_N are given functions, k_1, k_2 are some constants. Assume that the problem has a unique solution in the class of sufficiently smooth functions.

Let $V = \{v\in H^1(0,T;H^1(\Omega)) : v|_{\Gamma_D} = g_D\}$, $V_0 = \{v\in H^1(\Omega) : v|_\Gamma = 0\}$.

Define a weak statement of the problem (16), (18): find $s\in V$ such that for all $w\in V_0$:

$$(\partial_t s, w) + (k_1\nabla s, \nabla w) + (\vec{u}\cdot\nabla s, w) + (k_2\nabla\partial_t s, \nabla w) = (f, w), \tag{19}$$

where $(\cdot,\cdot)$ denotes the scalar product in $L^2(\Omega)$.

Stabilized methods. Introduce the quasi-uniform triangulation Θ in Ω and let N_e be the number of elements in Θ . Let $V_h \subset V$ be the finite element space defined as follows:

$$V_h = \{w\in V,\ v|_K \in P_1(K)\ \forall K\in\Theta\},$$

where $P_k(K)$ is the space of polynomials of degree at most k on the triangle K. Introduce a uniform partition of the time interval [0, *T*] by points $t_n = n\tau$, $N\tau = T$, $\tau > 0$. Denote a finite dimensional approximation of *s* at $t = t_n$ by s_h^n . Then the standard Galerkin method for (16)-(18) is defined as follows. Let $s_h^{n-1}\in V_h$ be known. Find $s_h^n\in V_h$, such that

$$\left(s_h^n - s_h^{n-1}, w_h\right) + \tau a\left(s_h^n, w_h\right) + b\left(s_h^n - s_h^{n-1}, w_h\right) = \tau\varphi\left(w_h\right),\ \forall w_h \in V_h, \tag{20}$$

where

$$a\left(s_h, w_h\right) = \left(k\nabla s_h, \nabla w_h\right) + \left(\vec{v}\cdot\nabla s_h, w_h\right),\ b\left(s_h, w_h\right) = \left(k\nabla s_h, \nabla w_h\right),$$

$$\varphi\left(w_h\right) = \left(f, w_h\right) + \int_{\Gamma_N} w_h g_N \, d\sigma.$$

In previous studies of the authors, the equation (19) was solved by a combined finite volume element method and the following result was obtained.

Theorem 1. If the condition $f \in L^2\left(Q_T\right)$ holds and τ is sufficiently small, there exists a unique sequence of solutions s_h^n, $n = 1, 2, ..., N$ such that

$$\left\|s_h^n\right\|_0 + \tau\sqrt{\frac{k_1}{2T}}\left\|\nabla s_h^n\right\|_0 \le \max\left\{\sqrt{3}, \sqrt{3k_1L} + \tau\sqrt{\frac{k_1}{2T}}\right\}\left\|f^n\right\|_1. \tag{21}$$

Theorem 2. Let s be the solution to (19) and s_h^n be the sequence of solutions to (20). Then under the condition $N_0\nu^{-1}\left\|\vec{v}\right\|_0 \le \frac{1}{4}$, there exists $\tau_0 > 0$ such that for $\tau \le \tau_0$

$$\left\|s\left(t_n\right) - s_h^n\right\|_0 + c_1\tau\left\|s\left(t_n\right) - s_h^n\right\|_1 \le C\left(\tau^2 + h^2\right).$$

In this paper, we study the stabilized finite element methods for solving (20). The main class of these methods is based on an extension of the discrete variational formulation using a grid-dependent stabilization term. A general view of these methods for (20) is defined as follows: find $s_h \in V_h$ such that

$$\left(s_h^n - s_h^{n-1}, w_h\right) + \tau a\left(s_h^n, w_h\right) + b\left(s_h^n - s_h^{n-1}, w_h\right) + \tau S(s_h^n, w_h) = \tau\varphi\left(w_h\right),\ \forall w \in V_h,$$

where **S**(-,-) is the stabilizing term added to the standard Galerkin formulation, whose general form is

$$S(s_h, w_h) = \sum_K \tau_K\left(As_h - \varphi, \tilde{A}w_h\right)_K,$$

τ_K is the stabilization parameter. The specific choice of the operator $\tilde{A}$ leads to different stabilized methods. For example [16],

$$\begin{aligned} &\text{SUPG:} && \tilde{A}w_h = \vec{v}\cdot\nabla w_h, \\ &\text{GLS:} && \tilde{A}w_h = -k\Delta w_h + \vec{v}\cdot\nabla w_h, \\ &\text{USFEM:} && \tilde{A}w_h = k\Delta w_h + \vec{v}\cdot\nabla w_h. \end{aligned} \tag{22}$$

One of the important points in the implementation of stabilized methods is the choice of the stabilization parameter τ_K. The parameter is selected based on the properties of the problem, such as the discrete maximum principle, convergence analysis, stability, and others. The examples of stabilizing parameters are as follows [17-20]:

$$\tau_K^c = \left(\frac{4k_1}{h_K^2} + \frac{2|\vec{v}_K|}{h_K} \right)^{-1}, \qquad \tau_K^s = \left(9\left(\frac{4k_1}{h_K^2} \right)^2 + \left(\frac{2|\vec{v}_K|}{h_K} \right)^2 \right)^{-\frac{1}{2}},$$

$$\tau_K^a = \left(\frac{12k_1}{h_K^2} + \frac{2|\vec{v}_K|}{h_K} \right)^{-1}, \qquad \tau_K^{fv} = \left(\frac{6k_1}{h_K^2} \zeta\left(\frac{|\vec{v}_K| h_K}{3k_1} \right) \right)^{-1}, \tag{23}$$

where h_K is diameter of the element K, $\zeta(x) = \{1, 0 \le x \le 1; x, x \ge 1\}$.

Comparison of the stabilizing parameters. Let us compare the stabilized methods (22) and the stabilizing parameters (23) based on two computational experiments. The first computational experiment is to estimate the deviation from the upper and lower bounds of the solution using stabilized methods and stabilizing parameters and different grid configurations. The second computational experiment is to compare an approximate solution with a known exact solution.

Problem 1. In $Q_T = \Omega \times (0,T)$, where $\Omega = (0,1)\times(0,1)$ consider the equation (16) with the parameters $T = 1$, $\vec{v} = (0.15,1)$, $k_1 = 10^{-4}$, and initial and boundary conditions

$$s(x,0) = 0, \qquad s(x,t) = 1,\ x \in \{x_1 = 0\};\ s(x,t) = 0,\ x \in \{x_2 = 0\} \cup \{x_1 = 1\},$$

$$\nabla s \cdot \vec{n} = 0,\ x \in \{x_2 = 1\}.$$

In the computational experiment, three grid configurations containing 968, 3744, and 15110 elements were used. The value of the parameter τ is set to 10^{-2}. The calculations were performed until the time layer $n = 100$, corresponding to the time value $T = 1$, was achieved.

Implementation of (20) without the use of stabilization leads to the appearance of non-physical oscillations with an approach to the final time the absolute value of which is more than 60%. Table 1 illustrates the dependence of the deviation of the approximate solution on the exact bounds, depending on the stabilization method and the stabilizing parameters. The use of stabilization allows one to extinguish non-physical oscillations. In general, the deviation from the exact bounds is less than 3% for all three methods considered.

Table 1 **–** Results of the computational experiments for Problem 1

N_e	Exact bounds		Without stabilization		SUPG + τ_c		GLS + τ_c		USFEM + τ_c	
	min	max	min	max	min	max	min	max	min	max
968	0.00	1.00	-0.34	1.63	-0.03	1.03	-0.02	1.02	-0.02	1.03
3744	0.00	1.00	-0.26	1.28	-0.02	1.01	-0.01	1.00	-0.01	1.01
15110	0.00	1.00	-0.17	1.13	-0.01	1.00	-0.01	1.00	0.00	1.01

Problem 2. In $Q_T = \Omega\times(0,T)$, where $\Omega = (-0.5,0.5)\times(-0.5,0.5)$ the problem (16)-(18) with the parameters $k_1 = 10^{-4}$, $\vec{v} = \left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ and the right-hand side

$$f(x,t) = -\frac{1}{\sqrt{2}\varepsilon}\phi^2(x,t,\varepsilon)+\frac{1}{2\varepsilon}\phi^2(x,t,\varepsilon)-\frac{1}{\varepsilon^2}k_1\phi^2(x,t,\varepsilon)\tanh\left(\frac{x_1+x_2-t}{2\varepsilon}\right)-$$

$$-k_2\left[\frac{1}{\varepsilon^3}\phi^2(x,t,\varepsilon)\tanh\left(\frac{x_1+x_2-t}{2\varepsilon}\right)^2-\frac{1}{2\varepsilon^3}\phi^4(x,t,\varepsilon)\right]$$

is considered where $\phi(x,t,\varepsilon) = \operatorname{sech}\frac{x_1+x_2-t}{2\varepsilon}$ and $\varepsilon = 10^{-2}$. The exact solution of the problem is $s(x,t) = 0.5-\tanh\frac{x_1+x_2-t}{2\varepsilon}$.

The accuracy value was estimated in the L^2-norm. Two values of the τ parameter equal to 1/30 and 1/60 are accepted. The grid configuration was chosen in the same way as in Problem 1. To estimate the influence of the term with the third derivative of the solution, we considered two cases, $k_2 = 10^{-6}$ and $k_2 = 10^{-2}$. According to the results of computational experiments, the SUPG method was the most effective in the first case, and GLS and USFEM are in the second.

Table 2 – Results of the computational experiments for Problem 2, case $k_2 = 10^{-6}$

Methods	N_e	$\tau = 1/30$				$\tau = 1/60$			
		τ_K^c	τ_K^s	τ_K^a	τ_K^f	τ_K^c	τ_K^s	τ_K^a	τ_K^f
1	2	3	4	5	6	7	8	9	10
Without stabilization	968	1.2052	1.1246	1.0218	1.1089	1.1473	1.0045	0.9286	1.0083
	3744	0.8934	0.9924	0.8214	1.0086	0.9645	0.8531	0.6425	0.8028
	15110	0.6911	0.7512	0.6457	0.8645	0.6654	0.6491	0.4289	0.6732
SUPG	968	0.0515	0.0617	0.0654	0.0618	0.0143	0.0183	0.0192	0.0185
	3744	0.0212	0.0399	0.0347	0.0313	0.0071	0.0089	0.0093	0.0090
	15110	0.0091	0.0148	0.0098	0.0100	0.0033	0.0045	0.0047	0.0043
GLS	968	0.0623	0.0692	0.0649	0.0621	0.0147	0.0185	0.0188	0.0187
	3744	0.0325	0.0347	0.0332	0.0396	0.0079	0.0089	0.0090	0.0093
	15110	0.0112	0.0119	0.0113	0.0136	0.0032	0.0041	0.0043	0.0041
USFEM	968	0.0647	0.0645	0.0657	0.0589	0.0076	0.0189	0.0191	0.0190
	3744	0.0344	0.0375	0.0345	0.0315	0.0527	0.0095	0.0096	0.0095
	15110	0.0128	0.0112	0.0137	0.0132	0.0497	0.0043	0.0042	0.0044

Table 3 – Results of the computational experiments for Problem 2, case $k_2 = 10^{-2}$

Methods	N_e	τ = 1/30				τ = 1/60			
		τ_K^c	τ_K^s	τ_K^a	τ_K^f	τ_K^c	τ_K^s	τ_K^a	τ_K^f
1	2	3	4	5	6	7	8	9	10
Without stabilization	968	1.3211	1.2835	1.1256	1.2164	1.3858	1.2764	1.1014	1.4875
	3744	0.7547	0.8436	0.9182	0.9384	1.0645	0.8621	0.7574	1.0064
	15110	0.5583	0.6487	0.7365	0.6912	0.8257	0.6471	0.5314	0.6947
SUPG	968	0.0789	0.0808	0.0758	0.0718	0.0296	0.0287	0.0214	0.0273
	3744	0.0352	0.0438	0.0328	0.0394	0.0131	0.0114	0.0112	0.0117
	15110	0.0128	0.0200	0.0111	0.0182	0.0068	0.0051	0.0054	0.0058
GLS	968	0.0658	0.0687	0.0687	0.0654	0.0141	0.0114	0.0147	0.0146
	3744	0.0368	0.0341	0.0348	0.0323	0.0054	0.0047	0.0075	0.0052
	15110	0.0193	0.0187	0.0158	0.0187	0.0021	0.0024	0.0036	0.0027
USFEM	968	0.0618	0.0614	0.0625	0.0682	0.0747	0.0141	0.0116	0.0112
	3744	0.0384	0.0351	0.0387	0.0337	0.0674	0.0051	0.0057	0.0053
	15110	0.0161	0.0163	0.0178	0.0187	0.0131	0.0029	0.0025	0.0021

Conclusion. Thus, the application and comparison of the stabilized SUPG, GLS, and USFEM finite element methods for solving two-phase non-equilibrium flow problem have been investigated. The study showed the effectiveness of the methods considered. The results obtained will be used in subsequent studies.

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ЕКІ ФАЗАЛЫ ТЕПЕ-ТЕҢСІЗ ФИЛЬТРАЦИЯ ЕСЕБІНДЕГІ ҚАНЫҚТЫҚ ҮШІН ТЕҢДЕУДІ ШЕШУДІҢ ТҰРАҚТАНДЫРЫЛҒАН СОҢҒЫ ЭЛЕМЕНТТЕР ӘДІСІ

Бұл жұмыста екі фазалы тепе-теңсіз фильтрация есебіндегі қанықтық үшін теңдеуді шешудің жуықтау әдісі жасалды. Бұл есеп ҚР БҒМ қаржыландыратын AP08053189 жобасы *бойынша орындалатын зерттеулер шеңберінде қарастырылады. Бұл теңдеу конвекциясы басым болатын және шешімнің үшінші ретті туындысы қосымша кіретін конвекция-диффузия түріндегі теңдеуге жатады. Теңдеудің гиперболалық сипатына байланысты оны шешу бірқатар қиындықтар туғызады және шешу әдісін мұқият таңдау қажеттілігіне әкеледі. Қиындықтардың бірі - екі фазаның шекарасында физикалық емес тербелістердің пайда болуы. Есептеу тәжірибелеріне сүйене отырып, үш классикалық тұрақтандырылған ақырлы элементтер әдістері (SUPG, GLS және USFEM) салыстырылды. Сонымен қатар, тұрақтандырылған әдістердің осы параметрлерді таңдауға сезімталдығына және орнықтылық пен дәлдіктің оларға айтарлықтай тәуелділігіне байланысты бірнеше тұрақтандыру параметрлерін қолдана отырып салыстырмалы есептеулер жүргізілді. Әдістемелік есептеулер жүргізілді және тор конфигурациясы мен тұрақтандыру параметрлерінің әртүрлі мәндерін пайдаланып жүргізілген есептеулер нәтижелері ұсынылды.*

***Түйін сөздер**: ақырлы элементтер әдісі, тұрақтандырылған әдіс, тепе-теңсіз фильтрация, SUPG, GLS, USFEM.*

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СТАБИЛИЗИРОВАННЫЙ МЕТОД КОНЕЧНЫХ ЭЛЕМЕНТОВ ДЛЯ УРАВНЕНИЯ ДЛЯ НАСЫЩЕННОСТИ В ЗАДАЧЕ ФИЛЬТРАЦИИ ДВУХФАЗНОЙ НЕРАВНОВЕСНОЙ ЖИДКОСТИ

В данной работе построен приближенный метод решения уравнения для насыщенности в задаче двухфазной неравновесной фильтрации. Данная задача изучается в рамках исследований, выполняемых по проекту, финансируемому МОН РК, грант AP08053189. Это уравнение относится к уравнению типа конвекции-диффузии с преобладанием конвекции и с дополнительным членом, содержащим производную решения третьего порядка. Из-за гиперболического характера уравнения его решение сопровождается рядом трудностей, которые приводят к необходимости тщательного выбора метода решения. Одной из трудностей является появление нефизических осцилляций на границе раздела двух фаз. На основе вычислительных экспериментов проведено сравнение трех классических стабилизированных методов конечных элементов (SUPG, GLS и USFEM). Кроме того, проведены сравнительные расчеты с использованием нескольких параметров стабилизации в связи с чувствительностью стабилизированных методов к выбору данных параметров и значительной зависимостью стабильности и точности от них. Проведены методические расчеты и представлены результаты расчетов с различными значениями конфигураций сетки и параметров стабилизации.

***Ключевые слова:** метод конечных элементов, стабилизированный метод, неравновесная фильтрация, SUPG, GLS, USFEM.*