UDC 519.6
https://doi.org/10.47533/2020.1606-146X.163

D. R. BAIGEREYEV*, N. B. ALIMBEKOVA

Amanzholov University, Ust-Kamenogorsk, Kazakhstan

ANALYSIS OF THE NUMERICAL SOLUTION OF THE THREE-PHASE NONISOTHERMAL FLUID FLOW PROBLEM

This paper is devoted to the construction and study of the stability and convergence of a numerical method for solving the problem of three-phase non-isothermal fluid flow in porous media, taking into account capillary forces, which was carried out within the framework of the grant project No. AP08053189 provided by the Ministry of Education and Science of the Republic of Kazakhstan. The model under consideration describes the processes occurring in oil reservoirs during the production of heavy oil by the method of thermal steam stimulation of the reservoir. The formulation of the differential problem is based on the introduction of a change of variables called global pressure, which makes it possible to exclude the capillary pressure gradient from the pressure equation. For the numerical solution, a numerical scheme was constructed. An a priori estimate in the energy norm is obtained, which expresses the stability of the constructed scheme with respect to the initial data and the right-hand sides of the equations. The theorem on the convergence of the solution of a numerical scheme to the solution of a differential problem is presented.

Keywords: *three-phase non-isothermal fluid flow, global pressure, convergence, stability.*

Introduction. The urgency of solving the problem of three-phase fluid flow in porous media is due to its important practical significance in predicting the production of high-viscosity paraffinic and highly viscous oil by steam injection into the reservoir. This is due to the fact that, at present, the reserves of this category of oil are several times higher than the reserves of so-called light oils, which leads to the need to use secondary or tertiary methods. However, due to the rather high cost of these methods, research aimed at improving its effectiveness is of great practical importance. At present, this can only be done by methods of mathematical modeling of fluid dynamics processes occurring in oil reservoirs during field development.

The mathematical model of three-phase non-isothermal fluid flow in porous media studied in this paper is a generalization of the two-phase isothermal fluid flow model constructed in [1, 2]. A lot of works are devoted to the study of the well-posedness of multiphase fluid flow problems with various assumptions about physical data, as well as the development and justification of computational algorithms for their approximate solution [3-7]

Numerous works are devoted to the numerical solution of three-phase fluid flow in porous media models [8, 9]. In [10], a numerical study of a multiphase flow model based on the application of the finite element method was carried out using the example of an underground hydrology problem. In [11], a method for solving the problem of a three-phase compositional model of compressible fluid flow is proposed, which combines a high-order discontinuous Galerkin method and a multiscale hybrid finite element method. [9] uses the Galerkin's modified weighted residual finite element method with asymmetric basis functions. The papers [12, 13] proposed an original high-order multiscale scheme

* E-mail корреспондирующего автора: d.baigereyev@gmail.com

for the incompressible case. In [4], using the example of model problems of three-phase isothermal fluid flow, it is shown that from a computational point of view, the solution of this problem using global pressure is more efficient than the solution of the problem in the phase formulation.

A review of the literature showed that no studies have been carried out in which the idea of introducing a global pressure is used to model the nonisothermal fluid flow of a three-phase compressible fluid. In this regard, in this paper, we propose a new formulation of the problem, which is based on a change of variables, similar to the original works [7, 14]. Following these papers, this change of variables is called the global pressure, and the resulting problem is the problem of three-phase non-isothermal fluid flow in the global formulation.

In [15] a three-phase non-isothermal fluid flow model was developed using the concept of global pressure, efficient difference schemes for its implementation were constructed, and theoretical studies of approximation, stability, and convergence were carried out. Further, in [16], cost-effective difference schemes for a particular case of this model are proposed and mathematically justified. The complexity of solving the problem is related to the complexity of the physical process, phase transitions, the need to track the position and characteristics of the thermal front with good accuracy, and the strong dependence of the solution accuracy on the grid cell size.

In this paper, the problem of three-phase non-isothermal fluid flow in porous media is considered taking into account capillary forces. An a priori estimate is obtained that expresses the stability of the scheme with respect to the initial data and the right-hand sides of the equations. The convergence of the solution of a difference scheme to the solution of a differential problem is presented.

Research methodology and results. In $Q=\overline{\Omega}\times[0,t_1]$, where $\overline{\Omega}=[0,l]\times[0,l]$, $l>0$, the problem of three-phase non-isothermal fluid flow in porous media of immiscible fluids in a homogeneous, isotropic medium is considered, with the capillary forces and phase transitions between the phases of water and coolant taken into consideration [15]:

$$c_T\frac{\partial T}{\partial t}+\vec{u}\cdot\nabla T-k_h\nabla^2T=f_T, \tag{1}$$

$$\beta_p\frac{\partial p}{\partial t}-\nabla\cdot\left(k_p(x,t,p)\nabla p\right)-\beta_T\frac{\partial T}{\partial t}+\nabla\cdot\left(k_T\nabla T\right)=f_p, \tag{2}$$

$$\frac{\partial s_w}{\partial t}-\nabla\cdot\left(a_w(x,t,s_w)\nabla s_w\right)-\nabla\cdot\left(\nu_w(x,t)\nabla(p-p_c)\right)=f_w, \tag{3}$$

$$\frac{\partial s_g}{\partial t}-\nabla\cdot\left(a_g(x,t,s_g)\nabla s_g\right)-\nabla\cdot\left(\nu_g(x,t)\nabla(p-p_c)\right)=f_g, \tag{4}$$

$$\vec{u}=-k\lambda(\gamma\nabla p-\xi\nabla T), \tag{5}$$

$$T(x,0)=T_0,\ p(x,0)=p_0,\ s_\alpha(x,0)=s_{\alpha 0}, \tag{6}$$

$$-k_h\frac{\partial T}{\partial x_m}=0,\ x_m=0;\ \ k_h\frac{\partial T}{\partial x_m}=0,\ x_m=l, \tag{7}$$

$$-k_p \frac{\partial p}{\partial x_m} = 0,\ x_m = 0;\ k_p \frac{\partial p}{\partial x_m} = 0,\ x_m = l, \tag{8}$$

$$-\nu_w \frac{\partial s_w}{\partial x_m} = 0,\ x_m = 0;\ \nu_w \frac{\partial s_w}{\partial x_m} = 0,\ x_m = l. \tag{9}$$

Here the subscripts w,o,g,r denote the phases of water, oil, coolant and rock; ϕ and k are the porosity and permeability of the medium; $p_\alpha(x,t)$ is the pressure, $s_\alpha(x,t)$ is the saturation, $\rho_\alpha(p_\alpha,T)$ is the density, $k_\alpha(s_\alpha)$ is the relative phase permeability, $\mu_\alpha(T)$ is the viscosity, $i_\alpha(T)$ is the enthalpy, U_α is the internal energy of phase α; k_h is the thermal conductivity coefficient; q_α and q_T are source/sink functions and heat output; $\vec{u}_\alpha$ is the filtration rate vector; I_α is the intensity of phase transitions.

Let us assume that the function k_p depends on spatial variables, time and global pressure; the functions $k_p = k_p(x,t,p)$, $k_h(x,t)$ and $\lambda(x,t)$ are continuous in $\overline{\Omega}\times[0,t_1]$ and the following conditions holds:

$$c_0 \le |k_p(p)| \le c_1,\ k_h \ge 4c_0,\ |\lambda| \le c_1,\ c_0, c_1 > 0. \tag{10}$$

Assume that the functions γ, ξ are calculated for some average values of pressure, temperature, and saturations and are known functions of the spatial variable and time, and

$$c_0 \le (\gamma, \xi) \le c_1. \tag{11}$$

Regarding the functions ν_α and a_α, we assume that $a_\alpha = a_\alpha(x,t,s_\alpha)$, $a_w = \nu_w = 0$ when $s_w = 0$; $a_g = \nu_g = 0$ for $s_g = 1$, and the following inequalities hold:

$$(a_w, a_g, \nu_\alpha) \le c_1, \tag{12}$$

The values c_T, k_h, β_p are assumed to be constant. Since the constants c_T, β_p do not affect the stability and convergence of the numerical scheme, we will assume that these constants are equal to one to simplify calculations, i.e. $c_T \equiv 1$, $\beta_p \equiv 1$. In addition, we assume that the function $\beta_T = \beta_T(x,t)$ and the constant k satisfy the inequalities

$$\beta_T \le c_2\tau, \tag{13}$$

$$k \le c_3\tau, \tag{14}$$

where τ is the time discretization parameter. Let function p_c be known, for which the following relation holds:

$$\nabla p_c = b_1\nabla s_w + b_2\nabla s_g + b_3\nabla p + b_4\nabla T, \tag{15}$$

where $b_1 = \dfrac{\partial p_c}{\partial s_w}, b_2 = \dfrac{\partial p_c}{\partial s_g}, b_3 = \dfrac{\partial p_c}{\partial p}, b_4 = \dfrac{\partial p_c}{\partial T}$, and

$$\left|b_1\right| \geq c_4 > 0, \left|b_2\right| \geq c_4 > 0, \left|b_3\right| \leq c_5, \left|b_4\right| \leq c_5. \tag{16}$$

Let us consider the following discrete scheme for Problem (1)-(5):

$$BT_t^h + L\left(\vec{u}^h, \widehat{T}^h\right) + \Lambda_1 T^h = f_T^h, \tag{17}$$

$$Bp_t^h + \Lambda_2 p^h = \beta_T T_t^h + \Lambda_9 T^h + f_p^h, \tag{18}$$

$$Bs_{w,t}^h + \Lambda_{3w} s_w^h + \Lambda_{5w} s_g^h + \Lambda_{7w} p^h + \Lambda_{8w} T^h = f_w^h, \tag{19}$$

$$Bs_{g,t}^h + \Lambda_{4g} s_g^h + \Lambda_{6g} s_w^h + \Lambda_{7g} p^h + \Lambda_{8g} T^h = f_g^h, \tag{20}$$

$$u_m^h = -k\lambda\left(\gamma^h p_{\bar{x}_m}^h + \xi^h T_{\bar{x}_m}^h\right), \tag{21}$$

$$T^h(0) = T_0,\ p^h(0) = p_0,\ s_\alpha^h(0) = s_{\alpha 0}, \tag{22}$$

where

$$B = E + \tau\omega A,\ \ A = A_1 + A_2,$$

$$L(\vec{u},\theta) = 0.5\sum_{m=1}^{2}\left(\beta_m^+(x) u_m^{+1_m}\theta_{x_m} + \beta_m^-(x) u_m \theta_{\bar{x}_m}\right),$$

$$\Lambda_1 = \sum_{m=1}^{2}\Lambda_{1,m},\ \Lambda_2 y = \sum_{m=1}^{2}\left(\chi_m^+(x)\Lambda_2^+ y + \chi_m^-(x)\Lambda_2^- y\right),$$

$$A_m y = \left\{-2h_m^{-1} y_{x_m} + y, x_m = 0; -y_{\bar{x}_m x_m} + y, x_m \in \Omega_{h,m}; 2h_m^{-1} y_{\bar{x}_m} + y, x_m = l\right\},$$

$$\Lambda_{1,m} y = \left\{-2h_m^{-1} k_h^h y_{x_m}, x_m = 0; -k_h^h y_{\bar{x}_m x_m}, x_m \in \Omega_{h,m}; 2h_m^{-1} k_h^h y_{\bar{x}_m}, x_m = l\right\},$$

$$\Lambda_{2,m}^+ y = \left\{-2h_m^{-1} k_p^h y_{x_m}, x_m = 0; -\left(k_p^h y_{x_m}\right)_{\bar{x}_m}, x_m \in \Omega_{h,m}; 2h_m^{-1}\left(k_p^h y_{x_m}\right)^{-1_m}, x_m = l\right\},$$

$$\Lambda_{2,m}^- y = \left\{-2h_m^{-1}\left(k_p^h y_{\bar{x}_m}\right)^{+1_m}, x_m = 0; -\left(k_p^h y_{\bar{x}_m}\right)_{x_m}, x_m \in \Omega_{h,m}; 2h_m^{-1} k_p^h y_{\bar{x}_m}, x_m = l\right\},$$

$$\Lambda_{3\alpha,m} y = \left\{-2h_m^{-1} \nu_\alpha^h y_{x_m}, x_m = 0; -\left(\nu_\alpha^h y_{\bar{x}_m}\right)_{x_m}, x_m \in \Omega_{h,m}; 2h_m^{-1} \nu_\alpha^h y_{\bar{x}_m}, x_m = l\right\},$$

$$\Lambda_4 y = \sum_{m=1}^{2}\left[\chi_m^+(x)\Lambda_{4\alpha}^+ y + \chi_m^-(x)\Lambda_{4\alpha}^+ y\right],$$

$$\Lambda_{4\alpha,m}^+ y = \left\{-2h_m^{-1}a_\alpha^h y_{x_m}, x_m = 0; -\left(a_\alpha^h y_{x_m}\right)_{\bar{x}_m}, x_m \in \Omega_{h,m}; 2h_m^{-1}\left(a_\alpha^h y_{x_m}\right)^{-1_m}, x_m = l\right\},$$

$$\Lambda_{4\alpha,m}^- y = \left\{-2h_m^{-1}\left(a_\alpha^h y_{\bar{x}_m}\right)^{+1_m}, x_m = 0; -\left(a_\alpha^h y_{\bar{x}_m}\right)_{x_m}, x_m \in \Omega_{h,m}; 2h_m^{-1}a_\alpha^h y_{\bar{x}_m}, x_m = l\right\},$$

$$\Lambda_{9,m} y = \left\{-2h_m^{-1}k_T^h y_{x_m}, x_m = 0; -k_T^h y_{\bar{x}_m x_m}, x_m \in \Omega_{h,m}; 2h_m^{-1}k_T^h y_{\bar{x}_m}, x_m = l\right\},$$

$$f_T^h = f_T + O\left(h^2\right),\ f_p^h = f_p + O\left(h^2\right),$$

and the functions $\beta^{\pm}(x)$ and $\chi^{\pm}(x)$ are defined as follows:

$$\beta_m^+(x) = \left\{2, x_m = 0; 1, x_m \in \Omega_{h,m}; 0, x_m = l\right\},\ \beta_m^-(x) = 2 - \beta_m^+(x)$$

$$\chi_m^+(x) = \left\{1, x_{3-m} = 0; 0.5, x_{3-m} \in \Omega_{h,3-m}; 0, x_{3-m} = l\right\},$$

$$\chi_m^-(x) = \left\{0, x_{3-m} = 0; 0.5, x_{3-m} \in \Omega_{h,3-m}; 1, x_{3-m} = l\right\}.$$

Let us introduce the following seminorms:

$$\|u\|_8^2 = \frac{1}{2}\sum_{m=1}^{2}\left[\left(\nu_w, u_{\bar{x}_m}^2\right)_+ + \left(\nu_w, u_{x_m}^2\right)_-\right], \|u\|_9^2 = \frac{1}{2}\sum_{m=1}^{2}\left[\left(\nu_g, u_{\bar{x}_m}^2\right)_+ + \left(\nu_g, u_{x_m}^2\right)_-\right].$$

Let us assume that the following conditions hold with respect to the initial values of the global pressure and temperature:

$$c_1\eta - q_0 t_1 - \left\|T^0\right\|_B^2 - c_5\left\|p^0\right\|_B^2 \ge 0, \tag{23}$$

$$c_1\eta - q_0 t_1 - \left\|\tilde{T}^0\right\|_B^2 - c_5\left\|\tilde{p}^0\right\|_B^2 \ge 0, \tag{24}$$

where $\eta > 0$ is some real parameter, c_5 is a positive constant, q_0 is the constant, determined by the formula $q_0 = c_4\sum_{j=1}^{W_n}\sum_{x\in\bar{\Omega}_h} b_h(x)\left(p_{inj}^2 + T_{inj}^2\right)\delta_h\left(x - x_j^{(w)}\right)h_1 h_2$, c_4 is the constant given by $c_4 = c_1 \cdot \max\left\{\frac{4(\eta+2)}{3c_0\eta}, \frac{6c_5}{c_0}\right\}$.

To study the stability of the scheme (17)-(22), consider the problem with perturbed initial condition $\left(\tilde{T}_0, \tilde{p}_0, \tilde{s}_{\alpha 0}\right)$ and right-hand sides $\left(\tilde{f}_T, \tilde{f}_p, \tilde{f}_\alpha\right)$ and denote the corresponding solution of the perturbed problem by $\left(\tilde{T}^h, \tilde{p}^h, \tilde{s}_w^h, \tilde{s}_g^h\right)$. Further, denote

$$\theta = T^h - \tilde{T}^h,\ \psi_T = f_T^h - \tilde{f}_T^h,\ \pi = p^h - \tilde{p}^h,\ \psi_p = f_p^h - \tilde{f}_p^h,$$

$$\sigma_\alpha = s_\alpha^h - \tilde{s}_\alpha^h,\ \psi_\alpha = f_\alpha^h - \tilde{f}_\alpha^h,\ \alpha = w, g,$$

$$\zeta_m = u_m - \tilde{u}_m,\ m = 1,2,$$

and obtain the problem for $\theta, \pi, \sigma_\alpha, \vec{\zeta}$:

$$B\theta_t + L\left(\vec{u}^h, \hat{T}^h\right) - L\left(\vec{\tilde{u}}^h, \hat{\tilde{T}}^h\right) + \Lambda_1 \hat{\theta} = \psi_T, \tag{25}$$

$$B\pi_t + \Lambda_2 p^h - \Lambda_2 \tilde{p}^h = \beta_T^h \theta_t + \Lambda_9 \theta + \psi_p, \tag{26}$$

$$B\sigma_{w,t} + \Lambda_{3w} s_w^h - \Lambda_{3w} \tilde{s}_w^h + \Lambda_{5w} \sigma_g + \Lambda_{7w} \pi + \Lambda_{8w} \theta = \psi_w, \tag{27}$$

$$B\sigma_{g,t} + \Lambda_{4g} s_g^h - \Lambda_{4g} \tilde{s}_g^h + \Lambda_{6g} \sigma_w + \Lambda_{7g} \pi + \Lambda_{8g} \theta = \psi_g, \tag{28}$$

$$\zeta_m = -k\lambda\left(\gamma^h \pi_{\bar{x}_m} + \xi^h \theta_{\bar{x}_m}\right), \tag{29}$$

$$\theta(0) = \theta^0,\ \pi(0) = \pi^0,\ \sigma_\alpha(0) = \sigma_\alpha^0. \tag{30}$$

Study of the stability of the constructed scheme is based on the following lemmas which are proved similarly to [15].

Lemma 1. Under the conditions (10) , (12), the following estimates holds:

$$2\tau\left(\Lambda_1 w, \tilde{w}\right) \le \|w\|_1^2 + \frac{\tau^2}{\varepsilon}\|\tilde{w}\|_A^2 + \frac{\tau}{\varepsilon}\|\tilde{w}\|_B^2,\ \varepsilon > 0,$$

$$\left(\Lambda_1 w, w\right) \ge 4c_0 \|w\|_1^2,$$

$$2\tau\left(\Lambda_{3w} w, \tilde{w}\right) \le \|w\|_1^2 + \frac{\tau^2}{\varepsilon}\|\tilde{w}\|_A^2 + \frac{\tau}{\varepsilon}\|\tilde{w}\|_B^2,\ \varepsilon > 0.$$

Lemma 2. Under the conditions (12), the operator Λ_{4w} satisfies the inequalities

$$\left(\Lambda_{4w} w - \Lambda_{4w} \tilde{w}, w - \tilde{w}\right) \ge c_0 \|w - \tilde{w}\|_8^2,$$

$$2\tau\left|\left(\Lambda_{4w} w - \Lambda_{4w} \tilde{w}, z\right)\right| \le 2c_1\left(\varepsilon\|w - \tilde{w}\|_1^2 + \frac{\tau^2}{\varepsilon}\|z\|_A^2 + \frac{\tau}{\varepsilon\omega\delta}\|z\|_B^2\right),\ \varepsilon > 0.$$

Lemma 3. Under the conditions (10), (13), where $\omega > \omega_1$,

$$\omega_1 = \left(\frac{4c_1^2}{c_0} + \frac{c_1}{\varepsilon}\right)\left(1 + \frac{1}{\delta}\right) + \frac{c_2}{\varepsilon\delta} + \frac{2c_1}{c_0\delta} \tag{31}$$

the following estimate holds:

$$\left\|\hat{\pi}\right\|_B^2 + \frac{c_0\tau}{2}\|\pi\|_1^2 \le d_1(\tau)\|\pi\|_B^2 + \frac{2c_2\varepsilon_2\tau^3}{\delta}\|\theta_t\|_A^2 + d_2(\varepsilon)\tau\|\theta\|_1^2 + d_3(\tau)\left\|\psi_p\right\|_{A^{-1}}^2.$$

Lemma 4. Under the conditions (10), (13), $\omega > \omega_2$,

$$\omega_2 = \frac{c_2}{\varepsilon_2\delta} + \frac{c_0}{3\delta} + \left(\frac{6c_1^2}{c_0} + \frac{2}{\varepsilon_3}\right)\left(1 + \frac{1}{\delta}\right), \tag{32}$$

the following inequalities hold:

$$\left\|\hat{p}^h\right\|_B^2 + \tau\left(\frac{c_0}{3} - \frac{c_2}{\varepsilon_2} - \frac{2}{\varepsilon_3}\right)\left\|p^h\right\|_1^2 \le \left\|p^h\right\|_B^2 + \frac{2c_2\varepsilon_2\tau^3}{\delta}\left\|T_t^h\right\|_A^2 + 4\tau\varepsilon_3\left\|T^h\right\|_1^2 + \frac{6\tau}{c_0}\left\|f_p^h\right\|_{A^{-1}}^2, \tag{33}$$

$$\left\|\hat{\tilde{p}}^h\right\|_B^2 + \tau\left(\frac{c_0}{3} - \frac{c_2}{\varepsilon_2} - \frac{2}{\varepsilon_3}\right)\left\|\tilde{p}^h\right\|_1^2 \le \left\|\tilde{p}^h\right\|_B^2 + \frac{2c_2\varepsilon_2\tau^3}{\delta}\left\|\tilde{T}_t^h\right\|_A^2 + 4\tau\varepsilon_3\left\|\tilde{T}^h\right\|_1^2 + \frac{6\tau}{c_0}\left\|\tilde{f}_p^h\right\|_{A^{-1}}^2,$$

where ε_2 , $\varepsilon_3 > 0$.

Lemma 5. Under the conditions (10), (11), (14), (23), (24), $\omega > \omega_3$,

$$\omega_3 = \frac{c_1^2(\eta+2)}{8c_0}\left(1 + \frac{1}{\delta}\right) + \frac{c_0\eta}{\delta(\eta+2)} + \frac{c_2(\eta + 2 - 4c_2\eta)}{32\delta\eta}\left(\frac{2c_0\eta}{\eta+2} - \frac{c_0}{24}\right), \tag{34}$$

the following inequalities hold:

$$\left\|T^h\right\|_B^2 + \left\|p^h\right\|_B^2 + d_5\left\|p^h\right\|_1^2 + d_4\left\|T^h\right\|_1^2 \le c_1\eta,$$

$$\left\|\tilde{T}^h\right\|_B^2 + \left\|\tilde{p}^h\right\|_B^2 + d_5\left\|\tilde{p}^h\right\|_1^2 + d_4\left\|\tilde{T}^h\right\|_1^2 \le c_1\eta.$$

Lemma 6. Under the conditions (14), (23) the following inequality holds:

$$2\tau\left(L\left(u^h, T^h\right) - L\left(\tilde{u}^h, \tilde{T}^h\right), \hat{\theta}\right) \le d_9\eta + \frac{2c_1\eta\tau^2}{\varepsilon}\|\theta\|_B^2 + \frac{2c_1\eta\tau^3}{\varepsilon\delta}\|\theta_t\|_A^2.$$

Lemma 7. Under the conditions (14), (23), (24), $\omega > \omega_4$,

$$\omega_4 = \frac{c_1\left(c_1 + \delta(c_1+1) + 2c_1\eta\right)}{2c_0\delta}, \tag{35}$$

the following estimate is valid:

$$\left\|\hat{\theta}\right\|_B^2 + \tau^2(\omega - \omega_4)\|\theta_t\|_A^2 + 2\tau(4c_0 - c_1\varepsilon_1)\|\theta\|_1^2 \le$$

$$\le d_9\eta + d_{10}(\tau)\|\theta\|_B^2 + \frac{4c_0\tau}{c_1}\|\psi_T\|_{A^{-1}}^2.$$

Lemma 8. Under the conditions (12), (16), $\omega > \omega_5$,

$$\omega_5 = \frac{1}{(1-\varepsilon_1)\delta}\left(2c_1\varepsilon_1(1+\delta) + \frac{c_1(1 + c_6 + c_7)}{\varepsilon_2}\left(1 + \frac{\varepsilon_2\delta + \delta + \varepsilon_2}{\delta}\right)\right), \tag{36}$$

where ε_1 and ε_2 are determined from the conditions

$$2c_0 - 4c_1\varepsilon_1^{-1} - \left(1+\varepsilon_2\right)\left(1+c_6+c_7\right)\varepsilon_2^{-1} > 0, \qquad (37)$$

$$2c_6 - c_6\varepsilon_2 - 2c_1\varepsilon_1 - 2c_7\varepsilon_2^{-1} > 0, \qquad (38)$$

the following esimate holds:

$$\left\|\hat{\sigma}_w\right\|_B^2 + d_{11}\tau\left\|\sigma_w\right\|_8^2 + d_{12}\tau\left\|\sigma_g\right\|_8^2 \le$$

$$\le d_{13}\left\|\sigma_w\right\|_B^2 + d_{14}(\tau)\left\|\sigma_g\right\|_B^2 + d_{15}\tau\|\pi\|_1^2 + d_{16}\tau\|\theta\|_1^2 + d_{17}\tau\left\|\psi_w\right\|_{A^{-1}}^2 .$$

Lemma 9. Under the conditions (12), (16), $\omega > \omega_5$, where ω_5 is defined in Lemma 8, the following estimate holds:

$$\left\|\hat{\sigma}_g\right\|_B^2 + d_{18}\tau\left\|\sigma_g\right\|_9^2 + d_{19}\tau\left\|\sigma_w\right\|_9^2 \le$$

$$\le d_{20}\left\|\sigma_g\right\|_B^2 + d_{21}(\tau)\left\|\sigma_w\right\|_B^2 + d_{22}\tau\|\pi\|_1^2 + d_{23}\tau\|\theta\|_1^2 + d_{24}\tau\left\|\psi_g\right\|_{A^{-1}}^2 .$$

Let us present a stability theorem for a numerical scheme.

Theorem 1. Under the conditions (10)-(14), (16), (23), (24), $\omega > \omega_0$, $\omega_0 = \max_{i=\overline{1,5}}\omega_i + \varepsilon$, $\varepsilon > 0$, where ω_i are defined by (31), (32), (34)-(36), the numerical scheme (17)-(22) is stable with respect to the initial data and right-hand sides of the equations, and the following estimate holds:

$$\left\|\hat{\theta}\right\|_B^2 + \left\|\hat{\pi}\right\|_B^2 + \left\|\hat{\sigma}_w\right\|_B^2 + \left\|\hat{\sigma}_g\right\|_B^2 + d_{25}\tau\|\theta\|_1^2 + d_{26}\tau\|\pi\|_1^2 +$$

$$+d_{11}\tau\left\|\sigma_w\right\|_8^2 + d_{12}\tau\left\|\sigma_g\right\|_8^2 + d_{18}\tau\left\|\sigma_g\right\|_9^2 + d_{19}\tau\left\|\sigma_w\right\|_9^2 \le$$

$$\le d_9\eta + d_{10}(\tau)\|\theta\|_B^2 + d_1(\tau)\|\pi\|_B^2 + d_{27}(\tau)\left\|\sigma_w\right\|_B^2 + d_{28}(\tau)\left\|\sigma_g\right\|_B^2 +$$

$$+\frac{4c_0\tau}{c_1}\left\|\psi_T\right\|_{A^{-1}}^2 + d_3(\tau)\left\|\psi_p\right\|_{A^{-1}}^2 + d_{17}\tau\left\|\psi_w\right\|_{A^{-1}}^2 + d_{24}\tau\left\|\psi_g\right\|_{A^{-1}}^2 .$$

Proof. Combining the results of Lemmas 3, 7, 8 and 9, we obtain the inequality

$$\left\|\hat{\theta}\right\|_B^2 + \left\|\hat{\pi}\right\|_B^2 + \left\|\hat{\sigma}_w\right\|_B^2 + \left\|\hat{\sigma}_g\right\|_B^2 + \tau^2\left(\omega - \omega_4 - \frac{2c_2\varepsilon_2}{\delta}\right)\left\|\theta_t\right\|_A^2 +$$

$$+2\tau\left(4c_0 - c_1\varepsilon_1 - d_2(\varepsilon_3) - d_{16} - d_{23}\right)\|\theta\|_1^2 + \tau\left(\frac{c_0}{2} - d_{15} - d_{22}\right)\|\pi\|_1^2 +$$

$$+d_{11}\tau\left\|\sigma_w\right\|_8^2 + d_{12}\tau\left\|\sigma_g\right\|_8^2 + d_{18}\tau\left\|\sigma_g\right\|_9^2 + d_{19}\tau\left\|\sigma_w\right\|_9^2 \le$$

$$\le d_9\eta + d_{10}(\tau)\|\theta\|_B^2 + d_1(\tau)\|\pi\|_B^2 + \left(d_{13} + d_{21}(\tau)\right)\left\|\sigma_w\right\|_B^2 + \left(d_{14}(\tau) + d_{20}\right)\left\|\sigma_g\right\|_B^2 +$$

$$+\frac{4c_0\tau}{c_1}\left\|\psi_T\right\|_{A^{-1}}^2 + d_3(\tau)\left\|\psi_p\right\|_{A^{-1}}^2 + d_{17}\tau\left\|\psi_w\right\|_{A^{-1}}^2 + d_{24}\tau\left\|\psi_g\right\|_{A^{-1}}^2 .$$

Choosing ε_1 , ε_2 , ε_3 from the condition of positiveness of the coefficients of $\|\theta_t\|_A^2$, $\|\theta\|_1^2$, $\|\pi\|_1^2$, we arrive at the assertion of the theorem.

Now we study the convergence of the numerical scheme. Let (T, p, s_α, u) be the solution to problem (1)-(9), and $(T^h, p^h, s_\alpha^h, u^h)$ be the solution to the discrete problem (17)-(22). Then the error $\theta = T^h - T$, $\pi = p^h - p$, $\sigma_\alpha = s_\alpha^h - s_\alpha$, $\vec{\zeta} = \vec{u}^h - \vec{u}$ satisfy the problem

$$c_T \cdot B\theta_t + L(u^h, T^h) - L(u, T) + k_h \Lambda_1 \theta = \psi_T, \tag{39}$$

$$\beta_p \cdot B\pi_t + \Lambda_2 p^h - \Lambda_2 p = \beta_T \theta_t + k_T \Lambda_1 \theta + \psi_p, \tag{40}$$

$$B\sigma_{w,t} + \Lambda_{3w}\sigma_w + \Lambda_{5w}\sigma_g + \Lambda_{7w}\pi + \Lambda_{8w}\theta = \psi_w, \tag{41}$$

$$Bs_{g,t}^h + \Lambda_{4g}\sigma_g + \Lambda_{6g}\sigma_w + \Lambda_{7g}\pi + \Lambda_{8g}\theta = \psi_g, \tag{42}$$

$$\zeta_m = -k\lambda^h \left(\gamma^h \pi_{\bar{x}_m} + \xi^h \theta^h_{\bar{x}_m}\right), \tag{43}$$

$$\theta(0) = 0,\ \pi(0) = 0,\ \sigma_\alpha(0) = 0, \tag{44}$$

where $(\psi_T, \psi_p, \psi_\alpha)$ is the approximation error on the solution:

$$\psi_T = F_T - L(u, T) - k_h \Lambda_1 T - c_T \cdot BT_t,$$

$$\psi_p = F_p - \Lambda_2 p + \beta_T T_t - k_T \Lambda_1 T - \beta_p \cdot Bp_t,$$

$$\psi_w = F_w - \Lambda_{3w}s_w - \Lambda_{5w}s_g - \Lambda_{7w}p - \Lambda_{8w}T,$$

$$\psi_g = F_o - \Lambda_{4g}s_g + \Lambda_{6g}s_w + \Lambda_{7g}p + \Lambda_{8g}T.$$

Similarly to the results discussed in [17, 18], we prove a theorem on the convergence of a numerical scheme to solving problems:

Theorem 2. Under the conditions of Theorem 1, the solution of the discrete problem (17)-(22) converges to the solution of the differential problem (1)-(9) with the order $O(h^{3/2} + \tau)$.

Conclusion. Thus, a discrete scheme is proposed for the numerical solution of the problem of three-phase non-isothermal fluid flow in porous media, taking into account capillary forces. Using the method of energy inequalities, an a priori estimate is obtained that expresses the stability of the difference scheme with respect to the initial data and the right-hand sides of the equations. A theorem on the convergence of the numerical scheme to the solution of problems is presented. Based on the theoretical results, we conclude that the presented numerical scheme is efficient and can be used for efficient modeling of three-phase non-isothermal fluid flows in porous media. In further works, the results of numerical experiments will be presented.

The work was carried out with the financial support of the Ministry of Education and Science of the Republic of Kazakhstan (grant AP08053189, 2020-2022).

REFERENCES

1 Muskat M., Meres M. W. The flow of heterogeneous fluids through porous media // Physics. 1936. – No. 7. – p. 346-363.

2 Leverett V. C. Flow of oil-water mixtures through unconsolidated sands // Trans AIME. 1939. – No. 132. – p. 381-401.

3 Chen Z. Reservoir Simulation: Mathematical Techniques in Oil Recovery. – Philadelphia: SIAM, 2007. – 219 p.

4 Chen Z., Ewing R. Comparison of various formulation of three-phase flow in porous media // Journal of Computational Physics. 1997. – No. 132. – p. 362-373.

5 Bruining J., Van Duijn C.J. Traveling waves in a finite condensation rate model for steam injection // Computational Geosciences. 2006. – V. 10 – No. 4. – p. 373-387.

6 Wojcik W., Temirbekov N., Baigereyev D. Fractional flow formulation for threephase nonisothermal flow in porous media // Przeglad Elektrotechniczny. 2016. – V. 92 – No. 7. – p. 24–31.

7 Antoncev S. N., Monahov V. N. Kraevye zadachi dlja nekotoryh vyrozhdajushhihsja uravnenij mehaniki sploshnoj sredy / Novosibirskij gosudarstvennyj universitet. – 1977. – 48 p.

8 Falta R. W., Pruess K., Javandel I. Numerical modeling of steam injection for the removal of nonaqueous phase liquids from the subsurface // Water Resoures Research. 1992. – V. 28 – No. 2. – p. 433–449.

9 Kaddouri A., Ferguson W. J. A nonisothermal threephase pollutant transport model, theoretical formulation and numerical applications // Oil and Hydrocarbon Spills. 2002. – V. III. – p. 23–33.

10 Chen Z., Ewing R. Fully discrete finite element analysis of multiphase flow in groundwater hydrology // SIAM Journal of Numerical Analysis. 1997. – V. 34 – No. 6. – p. 2228–2253.

11 Moortgat J., Sun S., Firoozabadi A. Compositional modeling of threephase flow with gravity using higherorder finite element methods // Water Resoures Research. 2011. – V. 47 – No. W05511. – p. 1–26.

12 Niessner J., Helmig R. Multiscale modeling of threephase threecomponent processes in heterogeneous porous media // Advances in Water Resources. 2007. – V. 30 – No. 11. – p. 2309–2325.

13 Niessner J., Helmig R. Multiphysics modeling of flow and transport in porous media using a downscaling approach // Advances in Water Resources. 2009. – V. 32 – No. 6. – p. 845–850.

14 Chavent G., Jaffre J. Mathematical models and finite elements for reservoir simulation. – Elsevier, 1986. – 375 p.

15 Bajgereev D. R. Jekonomichnye raznostnye shemy dlja zadachi trehfaznoj neizotermicheskoj fil'tracii v peremennyh «global'noe davlenie, nasyshhennost', temperatura»: Ph. D. thesis / D. R. Bajgereev. – Almaty, 2017.

16 Zhumagulov B., Temirbekov N., Baigereyev D. Efficient difference schemes for the threephase nonisothermal flow problem // AIP Conference Proceedings. 2017. – V. 1880 – No. 060001. – P. 1–10.

17 Baigereyev D., Rakhmetullina Zh., Mukhamedova R. et al. Study of a difference scheme for a model threephase nonisothermal flow problem // Covmestnyj vypusk zhurnalov Vestnik VKGTU im. D. Serikbaeva i Vychislitel'nye tehnologii. 2018. – V. 1(2) – No. 3. – p. 55–66.

18 Zhumagulov B. T., Temirbekov N. M., Baigereyev D. R. Efficient difference schemes for the threephase nonisothermal flow problem // AIP Conference Proceedings. 2017. – V. 1880. – p. 060001–1 – 060001–10.

Д. Р. БАЙГЕРЕЕВ, Н. Б. АЛИМБЕКОВА

Сәрсен Аманжолов атындағы Шығыс Қазақстан университеті,
Өскемен, Қазақстан

ҮШ ФАЗАЛЫ ИЗОТЕРМАЛЫҚ ЕМЕС ФИЛЬТРАЦИЯ ЕСЕБІНІҢ САНДЫҚ ШЕШІМІН ТАЛДАУ

Мақала Қазақстан Республикасы Білім және ғылым министрлігінің гранттық қаржыландырылған AP08053189 жобасы аясында жүргізілетін капиллярлық күштерді ескере отырып, үш фазалы изотермалық емес фильтрация есебін шешудің сандық әдісін құруға және оның орнықтылығы мен жинақталуын зерттеуге арналған. Қарастырылып отырған модель мұнай қатпарына бу-жылу әсерінен ауыр мұнай өндіру кезінде мұнай қатпарында болатын үрдістерді сипаттайды. Дифференциалды есепті қою қысым теңдеуінен капиллярлық қысым градиентін алып тастауға мүмкіндік беретін глобальді қысым айнымалыларын ауыстыруға негізделген. Есепті шешу үшін сандық әдіс құрастырылды. Бастапқы мәліметтер мен теңдеулердің оң жақтарына сәйкес құрастырылған сұлбаның орнықтылығын білдіретін энергиялық нормада априорлық баға алынды. Сандық сұлба шешімінің дифференциалдық есеп шешіміне жинақтылығы туралы теорема ұсынылды.

***Түйін сөздер**: үш фазалы изотермалық емес фильтрация, глобальді қысым, жинақтылық, орнықтылық.*

Д. Р. БАЙГЕРЕЕВ, Н. Б. АЛИМБЕКОВА

Восточно-Казахстанский университет имени Сарсена Аманжолова,
Усть-Каменогорск, Казахстан

АНАЛИЗ ЧИСЛЕННОГО РЕШЕНИЯ ЗАДАЧИ ТРЕХФАЗНОЙ НЕИЗОТЕРМИЧЕСКОЙ ФИЛЬТРАЦИИ

Настоящая статья посвящена построению и исследованию устойчивости и сходимости численного метода решения задачи трехфазной неизотермической фильтрации с учетом капиллярных сил, разработанного в рамках грантового проекта Министерства образования и науки Республики Казахстан AP08053189. Рассматриваемая модель описывает процессы, протекающие в нефтяных пластах при добыче тяжелой нефти методом паротеплового воздействия на пласт. Постановка дифференциальной задачи основана на введении замены переменных глобального давления, позволяющей исключить градиент капиллярного давления из уравнения для давления. Для решения задачи построен численный метод. Получена априорная оценка в энергетической норме, выражающая устойчивость построенной схемы по начальным данным и правым частям уравнений. Представлена теорема о сходимости решения численной схемы к решению дифференциальной задачи.

***Ключевые слова**: трехфазная неизотермическая фильтрация, глобальное давление, сходимость, устойчивость.*