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AFFINE AUTOMORPHISMS OF THE UNIVERSAL MULTIPLICATIVE ENVELOPING ALGEBRA OF THE TWO-DIMENSIONAL LEFT-SYMMETRIC ALGEBRA WITH ZERO MULTIPLICATION

We construct the basis of the universal multiplicative enveloping algebra of the left-symmetric algebra. We also describe the affine automorphisms of the universal multiplicative enveloping algebra of the two-dimensional left-symmetric algebra with zero multiplication.
***Key words**: left-symmetric algebra, universal multiplicative enveloping algebra, automorphism.*

Introduction. An algebra A over an arbitrary field k with a bilinear product $x \cdot y$ is called *a left-symmetric algebra*, if the identity

$$(xy)z - x(yz) = (yx)z - y(xz) \tag{1}$$

is satisfied for any $x, y, z \in A.$ In other words, the associator $(x, y, z) = (xy)z - x(yz)$ is symmetric with respect to x and y i.e.,

$$(x, y, z) = (y, x, z).$$

If instead of the identity (1) the identity

$$(x, y, z) = (x, z, y)$$

is satisfied for any $x, y, z \in A,$ i.e., the associator (x, y, z) is symmetric with respect to y and z then an algebra A is called *a right symmetric algebra.*

The variety of left-symmetric algebras is *Lie-admissible*, i.e., every left-symmetric algebra A with respect to the operation $[x, y] = xy - yx$ is a Lie algebra. The identity (1) can be rewritten as

$$[x, y]z = x(yz) - y(xz).$$

Direct calculation gives the Jacobi identity

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0.$$

The Lie algebra constructed in this way will be denoted by $A^{(-)}$.

In 1994, D. Segal constructed the basis of the free left-symmetric algebra [1]. In the same work, he constructed the universal enveloping left-symmetric algebra of the Lie algebra and proved an analogue of the Poincare-Birkhoff-Witt theorem. Some properties of the basis and identities of right-symmetric algebras were studied by A. Dzhumadildaev [2, 3].

It is well known that automorphisms of the polynomial algebra $k[x, y]$ are tame [4, 5]. Moreover, the automorphism group $Aut(k[x, y])$ of this algebra admits the amalgamated free product structure [5, 6], i.e.,

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$$Aut(k[x, y]) = A *_C B,$$

where A is the affine automorphism subgroup, B is the triangular automorphism subgroup, and $C = A \cap B.$ Similar results hold for free associative algebras [6, 7], free Poisson algebras in characteristic zero [8]. Moreover, the automorphism groups of these algebras are isomorphic to the automorphism group of the polynomial algebra.

In 1964, P. Cohn [9] proved that all automorphisms of finitely generated free Lie algebras over an arbitrary field are tame. J. Lewin [10] extended this result for free algebras of Nielsen-Schreier varieties. Recall that the varieties of all nonassociative algebras [11], commutative and anticommutative algebras [12], Lie algebras [13, 14] and Lie superalgebras [15, 16] over a field are Nielsen-Schreier.

In the work [17] L. Makar-Limanov, D. Kozybaev, U. Umirbaev proved that the automorphisms of the free right-symmetric algebra of rank two are tame. A. Alimbaev, A. Naurazbekova, D. Kozybaev [18] showed that the automorphism group of this algebra admits the amalgamated free product structure. Using this structure, they also proved the linearizability of the reductive automorphism group and the triangulation of locally nilpotent derivations of the free right-symmetric algebra of rank two in the case of characteristic zero.

The automorphism groups of polynomial algebras [19, 20, 21] and free associative algebras [22, 23] in three variables over a field of characteristic zero cannot be generated by all elementary automorphisms, i.e., wild automorphisms exist.

Using the Grobner-Shirshov bases methods, D. Kozybaev, U. Umirbaev [24] constructed the basis of the universal multiplicative enveloping algebra of the right-symmetric algebra. In the same work, they proved an analogue of the Magnus embedding for right-symmetric algebras.

The paper is organized as follows. In section 2, we rewrite for the left-symmetric algebra the result of D. Kozybaev, U. Umirbaev [24] on the basis of the universal multiplicative enveloping algebra of the right-symmetric algebra. In section 3, we describe the affine automorphisms of the universal multiplicative enveloping algebra of the two-dimensional left-symmetric algebra with zero multiplication. The question of describing all automorphisms of this algebra remains open. Although it was easy to notice that the automorphism groups of the left and the right universal multiplicative enveloping algebras of the two-dimensional left-symmetric algebra with zero multiplication are tame.

The basis of the universal multiplicative enveloping algebra. Let A be a left-symmetric algebra over an arbitrary field k Denote by $U(A)$ the universal multiplicative enveloping algebra [25] of the algebra A and denote by $l_x, r_x \in U(A)$ the universal operators of left and right multiplications by x where $x \in A$ $(l_x(a) = xa, r_x(a) = ax).$ Recall that $U(A)$ is an associative algebra with 1 generated by the operators of left multiplication l_x and right multiplication r_x where $x \in A$ The identity (1) directly implies the defining relations of the algebra $U(A)$:

$$l_x l_y - l_y l_x = l_{[x,y]}, \quad r_x l_y - l_y r_x - r_x r_y + r_{yx} = 0, \quad x, y \in A. \tag{2}$$

The linear basis of the algebra is described by the following

Theorem 1. Let A be a left-symmetric algebra with linear basis $x_1, x_2, \ldots, x_k, \ldots$. Then the basis of the universal multiplicative enveloping algebra $U(A)$ of A consists of words of the form

$$l_{x_{j_1}} l_{x_{j_2}} \ldots l_{x_{j_t}} r_{x_{i_1}} r_{x_{i_2}} \ldots r_{x_{i_s}}, \tag{3}$$

where $j_1 \le j_2 \le \ldots \le j_t, \quad s, t \ge 0.$

Proof. By definition, $U(A)$ is the associative algebra with generators $r_{x_i}, l_{x_i}, i \ge 1,$ and defining relations

$$l_{x_j} l_{x_k} - l_{x_k} l_{x_j} - l_{[x_j, x_k]} = 0, \quad j > k, \tag{4}$$

$$r_{x_j} l_{x_k} - l_{x_k} r_{x_j} - r_{x_j} r_{x_k} + r_{x_k x_j} = 0. \tag{5}$$

Put

$$l_{x_1} < l_{x_2} < \ldots < l_{x_s} < \ldots < r_{x_1} < r_{x_2} < \ldots < r_{x_s} < \ldots.$$

Let u, ϑ be arbitrary associative words in the alphabet $l_{x_1}, l_{x_2}, \ldots, l_{x_s}, \ldots, r_{x_1}, r_{x_2}, \ldots$. Put $u < \vartheta$ if one of the following conditions is true:

1) $d_l(u) < d_l(\vartheta),$ where d_l is the length function in the variables l_{x_i};

2) $d_l(u) = d_l(\vartheta), \quad d(u) < d(\vartheta),$ where d is the general length function in the variables l_{x_i}, r_{x_j};

3) $d_l(u) = d_l(\vartheta), \quad d(f) = d(g),$ u precedes ϑ with respect to the lexicographic order from left to right.

With respect to the order <, the leading terms of the left-hand sides of equalities (4) and (5) are words of the form $l_{x_j} l_{x_k}$ $(j > k),$ $r_{x_i} l_{x_k}$ (for all i, k). Therefore, they form a composition [26; 53-55] with the bases $\omega_1 = l_{x_j} l_{x_k} l_{x_i}, \quad j > k > i$ and $\omega_2 = r_{x_i} l_{x_j} l_{x_k},$ for $j > k$. Compute these compositions (below the comparison $\equiv$ means equality module terms with leading terms $< \omega_i, \quad i = 1, 2$) [26]:

Case 1. $\omega_1 = l_{x_j} l_{x_k} l_{x_i}, \quad j > k > i$ We have

$$(l_{x_j} l_{x_k} - l_{x_k} l_{x_j} - l_{[x_j, x_k]}) l_{x_i} - l_{x_j} (l_{x_k} l_{x_i} - l_{x_i} l_{x_k} - l_{[x_k, x_i]})$$

$$= -l_{x_k} l_{x_j} l_{x_i} - l_{[x_j, x_k]} l_{x_i} + l_{x_j} l_{x_i} l_{x_k} + l_{x_j} l_{[x_k, x_i]}$$

$$\equiv -l_{x_k} (l_{x_i} l_j + l_{[x_j, x_i]}) - l_{[x_j, x_k]} l_{x_i} + (l_{x_i} l_{x_j} + l_{[x_j, x_i]}) l_{x_k} + l_{x_j} l_{[x_k, x_i]}$$

$$\equiv -l_{x_k} l_{x_i} l_j - l_{x_k} l_{[x_j, x_i]} - l_{[x_j, x_k]} l_{x_i} + l_{x_i} l_{x_j} l_k + l_{[x_j, x_i]} l_{x_k} + l_{x_j} l_{[x_k, x_i]}$$

$$\equiv -(l_{x_i} l_{x_k} + l_{[x_k, x_i]}) l_{x_j} + l_{x_i} (l_{x_k} l_{x_j} + l_{[x_j, x_k]}) - l_{x_k} l_{[x_j, x_i]} - l_{[x_j, x_k]} l_{x_i} + l_{[x_j, x_i]} l_{x_k} + l_{x_j} l_{[x_k, x_i]}$$

$$\equiv -l_{x_i}l_{x_k}l_{x_j} - l_{[x_k,x_i]}l_{x_j} + l_{x_i}l_{x_k}l_{x_j} + l_{x_i}l_{[x_j,x_k]} - l_{x_k}l_{[x_j,x_i]} - l_{[x_j,x_k]}l_{x_i} + l_{[x_j,x_i]}l_{x_k} + l_{x_j}l_{[x_k,x_i]}$$

$$\equiv l_{x_j}l_{[x_k,x_i]} - l_{[x_k,x_i]}l_{x_j} + l_{[x_j,x_i]}l_{x_k} - l_{x_k}l_{[x_j,x_i]} + l_{x_i}l_{[x_j,x_k]} - l_{[x_j,x_k]}l_{x_i}$$

$$\equiv l_{[x_j,[x_k,x_i]]} + l_{[[x_j,x_i],x_k]} + l_{[x_i,[x_j,x_k]]}$$

$$\equiv l_{[x_i,[x_j,x_k]]+[x_j,[x_k,x_i]]+[x_k,[x_i,x_j]]} \equiv l_0 \equiv 0.$$

Case 2. $\omega_2 = r_{x_i}l_{x_j}l_{x_k}, \quad j > k.$ We have

$$r_{x_i}(l_{x_j}l_{x_k} - l_{x_k}l_{x_j} - l_{[x_j,x_k]}) - (r_{x_i}l_{x_j} - l_{x_j}r_{x_i} - r_{x_i}r_{x_j} + r_{x_jx_i})l_{x_k}$$

$$\equiv -r_{x_i}l_{x_k}l_{x_j} - r_{x_i}l_{[x_j,x_k]} + l_{x_j}r_{x_i}l_{x_k} + r_{x_i}r_{x_j}l_{x_k} - r_{x_jx_i}l_{x_k}$$

$$\equiv -(l_{x_k}r_{x_i} + r_{x_i}r_{x_k} - r_{x_kx_i})l_{x_j} + l_{x_j}(l_{x_k}r_{x_i} + r_{x_i}r_{x_k} - r_{x_kx_i}) + r_{x_i}(l_{x_k}r_{x_j} + r_{x_j}r_{x_k} - r_{x_kx_j})$$

$$-r_{x_i}l_{[x_j,x_k]} - r_{x_jx_i}l_{x_k} \equiv -l_{x_k}r_{x_i}l_{x_j} - r_{x_i}r_{x_k}l_{x_j} + r_{x_kx_i}l_{x_j} + l_{x_j}l_{x_k}r_{x_i} + l_{x_j}r_{x_i}r_{x_k}$$

$$-l_{x_j}r_{x_kx_i} + r_{x_i}l_{x_k}r_{x_j} + r_{x_i}r_{x_j}r_{x_k} - r_{x_i}r_{x_kx_j} - r_{x_i}l_{[x_j,x_k]} - r_{x_jx_i}l_{x_k}$$

$$\equiv -l_{x_k}(l_{x_j}r_{x_i} + r_{x_i}l_{x_j} - r_{x_jx_i}) - r_{x_i}(l_{x_j}r_{x_k} + r_{x_k}r_{x_j} - r_{x_jx_k})$$

$$+(l_{x_k}r_{x_i} + r_{x_i}r_{x_k} - r_{x_kx_i})r_{x_j} + l_{x_j}l_{x_k}r_{x_i} + l_{x_j}r_{x_i}r_{x_k} + r_{x_i}r_{x_j}r_{x_k} - l_{x_j}r_{x_kx_i}$$

$$+r_{x_kx_i}l_{x_j} - r_{x_jx_i}l_{x_k} - r_{x_i}l_{[x_j,x_k]} - r_{x_i}r_{x_kx_j}$$

$$\equiv -l_{x_j}r_{x_i}r_{x_k} - r_{x_i}r_{x_j}r_{x_k} + r_{x_jx_i}r_{x_k} + l_{x_k}r_{x_jx_i} + r_{x_i}r_{x_jx_k} + l_{[x_j,x_k]}r_{x_i} - r_{x_kx_i}r_{x_j}$$

$$+r_{x_kx_i}l_{x_j} + l_{x_j}r_{x_i}r_{x_k} - l_{x_j}r_{x_kx_i} + r_{x_i}r_{x_j}r_{x_k} - r_{x_i}r_{x_kx_j} - r_{x_i}l_{[x_j,x_k]} - r_{x_jx_i}l_{x_k}$$

$$\equiv r_{[x_j,x_k]x_i} - r_{x_j(x_kx_i)} + r_{x_k(x_jx_i)} \equiv r_{(x_jx_k - x_kx_j)x_i - x_j(x_kx_i) + x_k(x_jx_i)}$$

$$\equiv r_{(x_jx_k)x_i - (x_kx_j)x_i - x_j(x_kx_i) + x_k(x_jx_i)} \equiv 0.$$

Consequently, the defining relations (4), (5) of the algebra $U(A)$ are closed under composition. Then, by Shirshov lemma [26], the basis of the universal multiplicative enveloping algebra $U(A)$ consists of words of the form (3). Theorem 1 is proved.

Denote by $RU(A)$ the right universal multiplicative enveloping algebra of the algebra A, i.e., subalgebra of the algebra $U(A)$ generated by the universal operators r_x, where $x \in A$. Similarly, we define the left universal multiplicative enveloping algebra $LU(A)$ as the subalgebra of the algebra $U(A)$ generated by the universal operators l_x, where $x \in A$.

Corollary 1. 1) Under the conditions of Theorem 1, words of the form

$$l_{x_{j_1}} l_{x_{j_2}} \ldots l_{x_{j_t}}, \tag{6}$$

where $j_1 \leq j_2 \leq \ldots \leq j_t, \quad t \geq 0,$ form the basis of the algebra $LU(A)$.

2) Algebra $LU(A)$ is the associative algebra with generators $l_x, \quad i \geq 1,$ and defining relations (4).

3) Under the conditions of Theorem 1, the algebra $RU(A)$ is the free associative algebra with free set of generators

$$r_{x_{i_1}} r_{x_{i_2}}, \ldots, r_{x_{i_s}}, \ldots. \tag{7}$$

Proof. By definition, $LU(A)$ is the subalgebra of the associative algebra $U(A)$ generated by the elements $l_x, \quad i \geq 1.$ Using relation (4), it is easy to show that any element of $LU(A)$ is linearly expressed by words of the form (6). Words of the form (6) are part of the basis (3), consequently, they are also linearly independent and form the basis of $LU(A)$. This implies the statement of the corollary. The statement 3) follows directly from Theorem 1. Corollary 1 is proved.

Affine automorphisms. Let k be an arbitrary field and let A_2 be a two-dimensional left-symmetric algebra over the field k with linear basis e, f and $ef = fe = 0.$ Then the universal multiplicative enveloping algebra $U(A_2)$ of A_2 is generated by the operators r_e, r_f, l_e, l_f and the relations (2) imply the defining relations of $U(A_2)$

$$l_x l_y = l_y l_x, \quad r_x l_y = l_y r_x + r_x r_y, \quad x \in A_2. \tag{8}$$

Let $e < f$. Then, by Theorem 1, the basis of the algebra $U(A_2)$ consists of words of the form

$$\underbrace{l_e \ldots l_e}_{s} \underbrace{l_f \ldots l_f}_{t} r_{x_{i_1}} r_{x_{i_2}} \ldots r_{x_{i_m}},$$

where $x_{i_1}, \ldots, x_{i_m} \in \{e, f\}, \quad s, t, m \geq 0.$

Theorem 2. Let A_2 be the two-dimensional left-symmetric algebra over an arbitrary field k with linear basis e, f and $ef = fe = 0.$ If φ is the linear automorphism of the universal multiplicative enveloping algebra $U(A_2)$ of A_2 then

$$\varphi(l_e) = \alpha l_e + \beta l_f,$$
$$\varphi(l_f) = \gamma l_e + \delta l_f,$$
$$\varphi(r_e) = \alpha r_e + \beta r_f,$$
$$\varphi(r_f) = \gamma r_e + \delta r_f,$$

where $\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \neq 0, \quad \alpha, \beta, \gamma, \delta \in k.$

Proof. For convenience, denote r_e, r_f, l_e, l_f by $a, b, a', b',$ respectively. By definition, $U(A_2)$ is the associative algebra with generators a, b, a', b' and defining relations

$$b'a' = a'b', \tag{9}$$

$$ab' = b'a + ab, \tag{10}$$

$$ba' = a'b + ba, \tag{11}$$

$$aa' = a'a + a^2, \tag{12}$$

$$bb' = b'b + b^2. \tag{13}$$

Let φ be a linear automorphism of $U(A_2)$ and

$$\varphi(a') = \alpha_1 a + \beta_1 b + \alpha_1' a' + \beta_1' b',$$
$$\varphi(b') = \alpha_2 a + \beta_2 b + \alpha_2' a' + \beta_2' b',$$
$$\varphi(a) = \alpha_3 a + \beta_3 b + \alpha_3' a' + \beta_3' b',$$
$$\varphi(b) = \alpha_4 a + \beta_4 b + \alpha_4' a' + \beta_4' b',$$

where

$$\begin{vmatrix} \alpha_1 & \beta_1 & \alpha_1' & \beta_1' \\ \alpha_2 & \beta_2 & \alpha_2' & \beta_2' \\ \alpha_3 & \beta_3 & \alpha_3' & \beta_3' \\ \alpha_4 & \beta_4 & \alpha_4' & \beta_4' \end{vmatrix} \neq 0. \tag{14}$$

Since φ is the automorphism of the algebra $U(A_2)$ it follows from (13) that

$$\varphi(b)\varphi(b') = \varphi(b')\varphi(b) + \varphi(b)^2,$$

i.e.,

$$\begin{aligned}&(\alpha_4 a + \beta_4 b + \alpha_4' a' + \beta_4' b')(\alpha_2 a + \beta_2 b + \alpha_2' a' + \beta_2' b')\\ &= (\alpha_2 a + \beta_2 b + \alpha_2' a' + \beta_2' b')(\alpha_4 a + \beta_4 b + \alpha_4' a' + \beta_4' b')\\ &+(\alpha_4 a + \beta_4 b + \alpha_4' a' + \beta_4' b')(\alpha_4 a + \beta_4 b + \alpha_4' a' + \beta_4' b').\end{aligned} \tag{15}$$

Taking into account the defining relations (9) – (13), from (15) the equality follows

$$\begin{aligned}&(\alpha_2\alpha_4 a^2 + \alpha_2'\alpha_4 aa' + \alpha_2\alpha_4' a'a + \alpha_2'\alpha_4' a'^2) + (\beta_2\beta_4 b^2 + \beta_2'\beta_4 bb' + \beta_2\beta_4' b'b + \beta_2'\beta_4' b'^2)\\ &= (\alpha_2\alpha_4 a^2 + \alpha_2\alpha_4' aa' + \alpha_2'\alpha_4 a'a + \alpha_2'\alpha_4' a'^2 + \alpha_4^2 a^2 + \alpha_4\alpha_4' aa' + \alpha_4\alpha_4' a'a + \alpha_4'^2 a'^2)\\ &+(\beta_2\beta_4 b^2 + \beta_2\beta_4' bb' + \beta_2'\beta_4 b'b + \beta_2'\beta_4' b'^2 + \beta_4^2 b^2 + \beta_4\beta_4' bb' + \beta_4\beta_4' b'b + \beta_4'^2 b'^2).\end{aligned}$$

Applying (12) – (13) to the last equality, we obtain

$$\begin{aligned}&(\alpha_2'\alpha_4 a'a + \alpha_2'\alpha_4 a^2 + \alpha_2\alpha_4' a'a) + (\beta_2'\beta_4 b'b + \beta_2'\beta_4 b^2 + \beta_2\beta_4' b'b)\\ &= (\alpha_2\alpha_4' a'a + \alpha_2\alpha_4' a^2 + \alpha_2'\alpha_4 a'a + \alpha_4^2 a^2 + 2\alpha_4\alpha_4' a'a + \alpha_4\alpha_4' a^2 + \alpha_4'^2 a'^2)\\ &+(\beta_2\beta_4' b'b + \beta_2\beta_4' b^2 + \beta_2'\beta_4 b'b + \beta_4^2 b^2 + 2\beta_4\beta_4' b'b + \beta_4\beta_4' b^2 + \beta_4'^2 b'^2).\end{aligned}$$

This equality can be written as

$$(\alpha_2\alpha_4' + \alpha_4^2 + \alpha_4\alpha_4' - \alpha_2'\alpha_4)a^2 + 2\alpha_4\alpha_4'a'a + \alpha_4'^2a'^2$$
$$+(\beta_2\beta_4' + \beta_4^2 + \beta_4\beta_4' - \beta_2'\beta_4)b^2 + 2\beta_4\beta_4'b'b + \beta_4'^2b'^2 = 0.$$

This equality implies the equality system

$$\begin{cases} \alpha_4' = \beta_4' = 0, \\ \alpha_4(\alpha_4 - \alpha_2') = 0, \\ \beta_4(\beta_4 - \beta_2') = 0. \end{cases} \tag{16}$$

Taking into account the defining relations (9) – (13), from (15) the equality follows

$$(\alpha_4\beta_2 ab + \alpha_4\beta_2'ab' + \alpha_2\beta_4 ba + \alpha_2'\beta_4 ba')$$
$$= (\alpha_2\beta_4 ab + \alpha_4\beta_2 ba + \alpha_2'\beta_4 a'b + \alpha_4\beta_2'b'a + \alpha_4\beta_4 ab + \alpha_4\beta_4 ba).$$

Applying (10) – (11) to the last equality, we obtain

$$(\alpha_4\beta_2 ab + \alpha_4\beta_2'b'a + \alpha_4\beta_2'ab + \alpha_2\beta_4 ba + \alpha_2'\beta_4 a'b + \alpha_2'\beta_4 ba)$$
$$= (\alpha_2\beta_4 ab + \alpha_4\beta_2 ba + \alpha_2'\beta_4 a'b + \alpha_4\beta_2'b'a + \alpha_4\beta_4 ab + \alpha_4\beta_4 ba).$$

This equality can be written as

$$(\alpha_2\beta_4 + \alpha_4\beta_4 - \alpha_4\beta_2 - \alpha_4\beta_2')ab + (\alpha_4\beta_2 + \alpha_4\beta_4 - \alpha_2\beta_4 - \alpha_2'\beta_4)ba = 0.$$

Hence

$$\begin{cases} \alpha_4(\beta_4 - \beta_2') - \alpha_4\beta_2 + \alpha_2\beta_4 = 0, \\ \beta_4(\alpha_4 - \alpha_2') + \alpha_4\beta_2 - \alpha_2\beta_4 = 0. \end{cases} \tag{17}$$

Since φ is the automorphism of the algebra $U(A_2)$ it follows from (12) that

$$\varphi(a)\varphi(a') = \varphi(a')\varphi(a) + \varphi(a)^2,$$

i.e.,

$$\begin{gathered} (\alpha_3 a + \beta_3 b + \alpha_3'a' + \beta_3'b')(\alpha_1 a + \beta_1 b + \alpha_1'a' + \beta_1'b') \\ (\alpha_1 a + \beta_1 b + \alpha_1'a' + \beta_1'b')(\alpha_3 a + \beta_3 b + \alpha_3'a' + \beta_3'b') \\ (\alpha_3 a + \beta_3 b + \alpha_3'a' + \beta_3'b')(\alpha_3 a + \beta_3 b + \alpha_3'a' + \beta_3'b'). \end{gathered} \tag{18}$$

Taking into account the defining relations (9) – (13), from (18) the equality follows

$$(\alpha_1\alpha_3 a^2 + \alpha_1'\alpha_3 aa' + \alpha_1\alpha_3'a'a + \alpha_1'\alpha_3'a'^2) + (\beta_1\beta_3 b^2 + \beta_1'\beta_3 bb' + \beta_1\beta_3'b'b + \beta_1'\beta_3'b'^2)$$
$$= (\alpha_1\alpha_3 a^2 + \alpha_1\alpha_3'aa' + \alpha_1'\alpha_3 a'a + \alpha_1'\alpha_3'a'^2 + \alpha_3^2 a^2 + \alpha_3\alpha_3'aa' + \alpha_3\alpha_3'a'a + \alpha_3'^2 a'^2)$$
$$+(\beta_1\beta_3 b^2 + \beta_1\beta_3'bb' + \beta_1'\beta_3 b'b + \beta_1'\beta_3'b'^2 + \beta_3^2 b^2 + \beta_3\beta_3'bb' + \beta_3\beta_3'b'b + \beta_3'^2 b'^2).$$

Applying (12) – (13) to the last equality, we obtain

$$\begin{aligned}&(\alpha_1'\alpha_3 a'a+\alpha_1'\alpha_3 a^2+\alpha_1\alpha_3' a'a)+(\beta_1'\beta_3 b'b+\beta_1'\beta_3 b^2+\beta_1\beta_3' b'b)\\&=(\alpha_1\alpha_3' a'a+\alpha_1\alpha_3' a^2+\alpha_1'\alpha_3 a'a+\alpha_3^2 a^2+2\alpha_3\alpha_3' a'a+\alpha_3\alpha_3' a^2+\alpha_3'^2 a'^2)\\&+(\beta_1\beta_3' b'b+\beta_1\beta_3' b^2+\beta_1'\beta_3 b'b+\beta_3^2 b^2+2\beta_3\beta_3' b'b+\beta_3\beta_3' b^2+\beta_3'^2 b'^2).\end{aligned}$$

This equality can be written as

$$\begin{aligned}&(\alpha_1\alpha_3'+\alpha_3^2+\alpha_3\alpha_3'-\alpha_1'\alpha_3)a^2+2\alpha_3\alpha_3' a'a+\alpha_3'^2 a'^2\\&+(\beta_1\beta_3'+\beta_3^2+\beta_3\beta_3'-\beta_1'\beta_3)b^2+2\beta_3\beta_3' b'b+\beta_3'^2 b'^2=0.\end{aligned}$$

Hence

$$\begin{cases}\alpha_3'=\beta_3'=0,\\ \alpha_3(\alpha_3-\alpha_1')=0,\\ \beta_3(\beta_3-\beta_1')=0.\end{cases} \tag{19}$$

(18) also implies the equality

$$\begin{aligned}&(\alpha_3\beta_1 ab+\alpha_3\beta_1' ab'+\alpha_1\beta_3 ba+\alpha_1'\beta_3 ba')\\&=(\alpha_1\beta_3 ab+\alpha_3\beta_1 ba+\alpha_1'\beta_3 a'b+\alpha_3\beta_1' b'a+\alpha_3\beta_3 ab+\alpha_3\beta_3 ba).\end{aligned}$$

Applying (10) – (11) to the last equality, we obtain

$$\begin{aligned}&(\alpha_3\beta_1 ab+\alpha_3\beta_1' b'a+\alpha_3\beta_1' ab+\alpha_1\beta_3 ba+\alpha_1'\beta_3 a'b+\alpha_1'\beta_3 ba)\\&=(\alpha_1\beta_3 ab+\alpha_3\beta_1 ba+\alpha_1'\beta_3 a'b+\alpha_3\beta_1' b'a+\alpha_3\beta_3 ab+\alpha_3\beta_3 ba).\end{aligned}$$

This can be written as

$$(\alpha_1\beta_3+\alpha_3\beta_3-\alpha_3\beta_1-\alpha_3\beta_1')ab+(\alpha_3\beta_1+\alpha_3\beta_3-\alpha_1\beta_3-\alpha_1'\beta_3)ba=0.$$

Hence

$$\begin{cases}\alpha_3(\beta_3-\beta_1')-\alpha_3\beta_1+\alpha_1\beta_3=0,\\ \beta_3(\alpha_3-\alpha_1')+\alpha_3\beta_1-\alpha_1\beta_3=0.\end{cases} \tag{20}$$

Taking into account (14) and the first equalities of systems (16) and (19), we obtain

$$\begin{vmatrix}\alpha_3 & \beta_3\\ \alpha_4 & \beta_4\end{vmatrix}\cdot\begin{vmatrix}\alpha_1' & \beta_1'\\ \alpha_2' & \beta_2'\end{vmatrix}\neq 0.$$

Consequently,

$$\begin{vmatrix}\alpha_3 & \beta_3\\ \alpha_4 & \beta_4\end{vmatrix}\neq 0,\quad \begin{vmatrix}\alpha_1' & \beta_1'\\ \alpha_2' & \beta_2'\end{vmatrix}\neq 0. \tag{21}$$

Considering that φ is the automorphism, it follows from (11) that

$$\varphi(b)\varphi(a')=\varphi(a')\varphi(b)+\varphi(b)\varphi(a),$$

i.e.,

$$(\alpha_4 a+\beta_4 b)(\alpha_1 a+\beta_1 b+\alpha_1' a'+\beta_1' b')$$
$$=(\alpha_1 a+\beta_1 b+\alpha_1' a'+\beta_1' b')(\alpha_4 a+\beta_4 b)+(\alpha_4 a+\beta_4 b)(\alpha_3 a+\beta_3 b).$$

Opening the brackets, we get

$$(\alpha_1\alpha_4 a^2+\alpha_1'\alpha_4 aa')+(\beta_1\beta_4 b^2+\beta_1'\beta_4 bb')+(\alpha_4\beta_1 ab+\alpha_4\beta_1' ab'+\alpha_1\beta_4 ba+\alpha_1'\beta_4 ba')$$
$$=(\alpha_1\alpha_4 a^2+\alpha_1'\alpha_4 a'a+\alpha_3\alpha_4 a^2)+(\beta_1\beta_4 b^2+\beta_1'\beta_4 b'b+\beta_3\beta_4 b^2)$$
$$+(\alpha_1\beta_4 ab+\alpha_4\beta_1 ba+\alpha_1'\beta_4 a'b+\alpha_4\beta_1' b'a+\alpha_4\beta_3 ab+\alpha_3\beta_4 ba).$$

Applying (10) – (13) to the last equality, we obtain

$$(\alpha_1'\alpha_4 a'a+\alpha_1'\alpha_4 a^2)+(\beta_1'\beta_4 b'b+\beta_1'\beta_4 b^2)$$
$$+(\alpha_4\beta_1 ab+\alpha_4\beta_1' b'a+\alpha_4\beta_1' ab+\alpha_1\beta_4 ba+\alpha_1'\beta_4 a'b+\alpha_1'\beta_4 ba)$$
$$=(\alpha_1'\alpha_4 a'a+\alpha_3\alpha_4 a^2)+(\beta_1'\beta_4 b'b+\beta_3\beta_4 b^2)$$
$$+(\alpha_1\beta_4 ab+\alpha_4\beta_1 ba+\alpha_1'\beta_4 a'b+\alpha_4\beta_1' b'a+\alpha_4\beta_3 ab+\alpha_3\beta_4 ba).$$

This equality can be written as

$$(\alpha_3\alpha_4-\alpha_1'\alpha_4)a^2+(\beta_3\beta_4-\beta_1'\beta_4)b^2$$
$$+(\alpha_1\beta_4+\alpha_4\beta_3-\alpha_4\beta_1-\alpha_4\beta_1')ab+(\alpha_4\beta_1+\alpha_3\beta_4-\alpha_1\beta_4-\alpha_1'\beta_4)ba=0.$$

This implies the following equality system

$$\begin{cases}\alpha_4(\alpha_3-\alpha_1')=0,\\ \beta_4(\beta_3-\beta_1')=0,\\ \alpha_4(\beta_3-\beta_1')-\alpha_4\beta_1+\alpha_1\beta_4=0,\\ \beta_4(\alpha_3-\alpha_1')+\alpha_4\beta_1-\alpha_1\beta_4=0.\end{cases} \tag{22}$$

Considering that φ is the automorphism, it follows from (10) that

$$\varphi(a)\varphi(b')=\varphi(b')\varphi(a)+\varphi(a)\varphi(b),$$

i.e.,

$$(\alpha_3 a+\beta_3 b)(\alpha_2 a+\beta_2 b+\alpha_2' a'+\beta_2' b')$$
$$=(\alpha_2 a+\beta_2 b+\alpha_2' a'+\beta_2' b')(\alpha_3 a+\beta_3 b)+(\alpha_3 a+\beta_3 b)(\alpha_4 a+\beta_4 b).$$

Opening the brackets, we get

$$(\alpha_2\alpha_3 a^2+\alpha_2'\alpha_3 aa')+(\beta_2\beta_3 b^2+\beta_2'\beta_3 bb')+(\alpha_3\beta_2 ab+\alpha_3\beta_2' ab'+\alpha_2\beta_3 ba+\alpha_2'\beta_3 ba')$$
$$=(\alpha_2\alpha_3 a^2+\alpha_2'\alpha_3 a'a+\alpha_3\alpha_4 a^2)+(\beta_2\beta_3 b^2+\beta_2'\beta_3 b'b+\beta_3\beta_4 b^2)$$
$$+(\alpha_2\beta_3 ab+\alpha_3\beta_2 ba+\alpha_2'\beta_3 a'b+\alpha_3\beta_2' b'a+\alpha_3\beta_4 ab+\alpha_4\beta_3 ba).$$

Applying (10) – (13) to the last equality, we obtain

$$(\alpha_2'\alpha_3 a'a + \alpha_2'\alpha_3 a^2) + (\beta_2'\beta_3 b'b + \beta_2'\beta_3 b^2)$$
$$+(\alpha_3\beta_2 ab + \alpha_3\beta_2' b'a + \alpha_3\beta_2' ab + \alpha_2\beta_3 ba + \alpha_2'\beta_3 a'b + \alpha_2'\beta_3 ba)$$
$$= (\alpha_2'\alpha_3 a'a + \alpha_3\alpha_4 a^2) + (\beta_2'\beta_3 b'b + \beta_3\beta_4 b^2)$$
$$+(\alpha_2\beta_3 ab + \alpha_3\beta_2 ba + \alpha_2'\beta_3 a'b + \alpha_3\beta_2' b'a + \alpha_3\beta_4 ab + \alpha_4\beta_3 ba).$$

This can be written as

$$(\alpha_3\alpha_4 - \alpha_2'\alpha_3)a^2 + (\beta_3\beta_4 - \beta_2'\beta_3)b^2$$
$$+(\alpha_2\beta_3 + \alpha_3\beta_4 - \alpha_3\beta_2 - \alpha_3\beta_2')ab + (\alpha_3\beta_2 + \alpha_4\beta_3 - \alpha_2\beta_3 - \alpha_2'\beta_3)ba = 0.$$

Hence

$$\begin{cases} \alpha_3(\alpha_4 - \alpha_2') = 0, \\ \beta_3(\beta_4 - \beta_2') = 0, \\ \alpha_3(\beta_4 - \beta_2') - \alpha_3\beta_2 + \alpha_2\beta_3 = 0, \\ \beta_3(\alpha_4 - \alpha_2') + \alpha_3\beta_2 - \alpha_2\beta_3 = 0. \end{cases} \tag{23}$$

It follows from (9) that

$$\varphi(a')\varphi(b') = \varphi(b')\varphi(a'),$$

i.e.,

$$(\alpha_1 a + \beta_1 b + \alpha_1' a' + \beta_1' b')(\alpha_2 a + \beta_2 b + \alpha_2' a' + \beta_2' b')$$
$$= (\alpha_2 a + \beta_2 b + \alpha_2' a' + \beta_2' b')(\alpha_1 a + \beta_1 b + \alpha_1' a' + \beta_1' b').$$

Opening the brackets, we get

$$(\alpha_1\alpha_2 a^2 + \alpha_1\alpha_2' aa' + \alpha_1'\alpha_2 a'a + \alpha_1'\alpha_2' a'^2) + (\beta_1\beta_2 b^2 + \beta_1\beta_2' bb' + \beta_1'\beta_2 b'b + \beta_1'\beta_2' b'^2)$$
$$+(\alpha_1\beta_2 ab + \alpha_1\beta_2' ab' + \alpha_2\beta_1 ba + \alpha_2'\beta_1 ba' + \alpha_1'\beta_2 a'b + \alpha_1'\beta_2' a'b' + \alpha_2\beta_1' b'a + \alpha_2'\beta_1' b'a')$$
$$= (\alpha_1\alpha_2 a^2 + \alpha_1'\alpha_2 aa' + \alpha_1\alpha_2' a'a + \alpha_1'\alpha_2' a'^2) + (\beta_1\beta_2 b^2 + \beta_1'\beta_2 bb' + \beta_1\beta_2' b'b + \beta_1'\beta_2' b'^2)$$
$$+(\alpha_2\beta_1 ab + \alpha_2\beta_1' ab' + \alpha_1\beta_2 ba + \alpha_1'\beta_2 ba' + \alpha_2'\beta_1 a'b + \alpha_2'\beta_1' a'b' + \alpha_1\beta_2' b'a + \alpha_1'\beta_2' b'a').$$

Applying (9) – (13) to the last equality, we obtain

$$(\alpha_1\alpha_2' a'a + \alpha_1\alpha_2' a^2 + \alpha_1'\alpha_2 a'a) + (\beta_1\beta_2' b'b + \beta_1\beta_2' b^2 + \beta_1'\beta_2 b'b)$$
$$+(\alpha_1\beta_2 ab + \alpha_1\beta_2' b'a + \alpha_1\beta_2' ab + \alpha_2\beta_1 ba + \alpha_2'\beta_1 a'b + \alpha_2'\beta_1 ba + \alpha_1'\beta_2 a'b + \alpha_2\beta_1' b'a)$$
$$= (\alpha_1'\alpha_2 a'a + \alpha_1'\alpha_2 a^2 + \alpha_1\alpha_2' a'a) + (\beta_1'\beta_2 b'b + \beta_1'\beta_2 b^2 + \beta_1\beta_2' b'b)$$
$$+(\alpha_2\beta_1 ab + \alpha_2\beta_1' b'a + \alpha_2\beta_1' ab + \alpha_1\beta_2 ba + \alpha_1'\beta_2 a'b + \alpha_1'\beta_2 ba + \alpha_2'\beta_1 a'b + \alpha_1\beta_2' b'a).$$

Write this equality as

$$(\alpha_1'\alpha_2 - \alpha_1\alpha_2')a^2 + (\beta_1'\beta_2 - \beta_1\beta_2')b^2$$
$$+(\alpha_2\beta_1 + \alpha_2\beta_1' - \alpha_1\beta_2 - \alpha_1\beta_2')ab + (\alpha_1\beta_2 + \alpha_1'\beta_2 - \alpha_2\beta_1 - \alpha_2'\beta_1)ba = 0.$$

Hence

$$\begin{cases} \alpha_1'\alpha_2 - \alpha_1\alpha_2' = 0, \\ \beta_1'\beta_2 - \beta_1\beta_2' = 0, \\ \alpha_2(\beta_1 + \beta_1') - \alpha_1(\beta_2 + \beta_2') = 0, \\ \beta_2(\alpha_1 + \alpha_1') - \beta_1(\alpha_2 + \alpha_2') = 0. \end{cases} \quad (24)$$

Let $\alpha_4 \neq 0, \beta_4 \neq 0.$ Then it follows from systems (16) and (22) that $\alpha_1' = \alpha_3, \beta_1' = \beta_3, \alpha_2' = \alpha_4, \beta_2' = \beta_4.$ Considering all of this, systems (17), (20), (22), (23) imply the equality system

$$\begin{cases} \alpha_2\beta_4 - \alpha_4\beta_2 = 0, \\ \alpha_1\beta_3 - \alpha_3\beta_1 = 0, \\ \alpha_1\beta_4 - \alpha_4\beta_1 = 0, \\ \alpha_2\beta_3 - \alpha_3\beta_2 = 0. \end{cases} \quad (25)$$

Multiplying both parts of the second equality of system (25) by β_4 and applying the third equality of the same system, we obtain

$$\beta_1(\alpha_4\beta_3 - \alpha_3\beta_4) = 0.$$

Analogically, multiplying both parts of the last equality of system (25) by α_4 and applying the first equality of this system, we obtain

$$\alpha_2(\alpha_4\beta_3 - \alpha_3\beta_4) = 0.$$

According to (21) $\alpha_4\beta_3 - \alpha_3\beta_4 \neq 0.$ Then it follows from the last two equalities that $\beta_1 = \alpha_2 = 0.$ Taking this into account, system (25) easily implies that $\alpha_1 = \beta_2 = 0.$

Let $\alpha_4 = 0, \beta_4 \neq 0$ or $\alpha_4 \neq 0, \beta_4 = 0.$ Then it follows from (21) that $\alpha_3 \neq 0$ or $\beta_3 \neq 0$, respectively. It's easy to see that systems (16), (17), (19), (20), (22), (23) imply $\alpha_1' = \alpha_3, \beta_1' = \beta_3,$ $\alpha_2' = \alpha_4, \beta_2' = \beta_4$ and $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0.$

It is also easy to notice that for $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$ all equalities of the system (24) are satisfied. Consequently, if φ is the linear automorphism of the algebra $U(A_2)$ then

$$\varphi(a') = \alpha_3 a' + \beta_3 b',$$
$$\varphi(b') = \alpha_4 a' + \beta_4 b',$$
$$\varphi(a) = \alpha_3 a + \beta_3 b,$$
$$\varphi(b) = \alpha_4 a + \beta_4 b,$$

where $\begin{vmatrix} \alpha_3 & \beta_3 \\ \alpha_4 & \beta_4 \end{vmatrix} \neq 0.$ Theorem 2 is proved.

Theorem 3. Let A_2 be the two-dimensional left-symmetric algebra over an arbitrary field k with linear basis e, f and $ef = fe = 0$. If φ is the affine automorphism of the universal multiplicative enveloping algebra $U(A_2)$ of A_2 then

$$\varphi(l_e) = \alpha l_e + \beta l_f + \nu,$$
$$\varphi(l_f) = \gamma l_e + \delta l_f + \mu,$$
$$\varphi(r_e) = \alpha r_e + \beta r_f,$$
$$\varphi(r_f) = \gamma r_e + \delta r_f,$$

where $\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \neq 0, \quad \alpha, \beta, \gamma, \delta, \nu, \mu \in k.$

Proof. As in the proof of Theorem 1 denote r_e, r_f, l_e, l_f by $a, b, a', b',$ respectively. Let φ be the affine automorphism of $U(A_2)$ Then, by Theorem 1,

$$\varphi(a') = \alpha a' + \beta b' + \nu_1,$$
$$\varphi(b') = \gamma a' + \delta b' + \nu_2,$$
$$\varphi(a) = \alpha a + \beta b + \nu_3,$$
$$\varphi(b) = \gamma a + \delta b + \nu_4.$$

It's easy to see that defining relations (9) – (13) imply the statement of this Theorem. Theorem 3 is proved.

Let $LU(A_2)$ and $RU(A_2)$ be the left and the right universal multiplicative enveloping algebras of A_2 respectively.

Corollary 2. 1) $LU(A_2)$ is the free associative commutative algebra with the free generators l_e, l_f over a field k;

2) $RU(A_2)$ is the free associative algebra with the free generators r_e, r_f over a field k.

Let M be some variety of linear algebras over a field k. Let $F_n = k\langle x_1, \ldots, x_n \rangle$ be the free algebra of the variety M with the set of free generators $\{x_1, \ldots, x_n\}$. Denote by $\mathrm{Aut}(\mathrm{F_n})$ the automorphism group of this algebra. The automorphism φ of the algebra F_n such that $\varphi(x_i) = f_i, \quad 1 \leq i \leq n, \quad f_i \in F_n,$ is denoted by

$$\varphi = (f_1, f_2, \ldots, f_n).$$

The automorphism

$$\sigma(i, \alpha, f) = (x_1, x_2, \ldots, x_{i-1}, \alpha x_i + f, x_{i+1}, \ldots, x_n),$$

where $0 \neq \alpha \in k, \quad f \in k\langle x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \rangle,$ is called elementary.

An automorphism represented as a composition of elementary automorphisms is called tame. Otherwise, it is called wild.

In 1953, van der Kalk [5] proved that the automorphisms of a polynomial algebra in two variables over an arbitrary field are tame. A similar result for the free associative algebras was obtained by L. Makar Limanov [7] and A.G. Cherniyakievich [6]. Moreover, the automorphism groups of these algebras are isomorphic.

Corollary 3. All automorphisms of the algebras $LU(A_2)$ and $RU(A_2)$ are tame. Moreover,

$$Aut(LU(A_2)) \cong Aut(RU(A_2)).$$

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НӨЛДІК КӨБЕЙТІНДІСІ БАР ЕКІ ӨЛШЕМДІ СОЛ-СИММЕТРИЯЛЫ АЛГЕБРАНЫҢ УНИВЕРСАЛДЫ МУЛЬТИПЛИКАТИВТІ ОРАУШЫ АЛГЕБРАСЫНЫҢ АФФИНДІ АВТОМОРФИЗМДЕРІ

Сол-симметриялы алгебраның универсалды мультипликативті ораушы алгебрасының базисі тұрғызылған. Нөлдік көбейтіндісі бар екі өлшемді сол-симметриялы алгебраның универсалды мультипликативті ораушы алгебрасының аффинді автоморфизмдері сипатталған.

***Түйін сөздер**: сол-симметриялы алгебра, универсалды мультипликативті ораушы алгебра, автоморфизм.*

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АФФИННЫЕ АВТОМОРФИЗМЫ УНИВЕРСАЛЬНОЙ МУЛЬТИПЛИКАТИВНОЙ ОБЕРТЫВАЮЩЕЙ АЛГЕБРЫ ДВУМЕРНОЙ ЛЕВОСИММЕТРИЧНОЙ АЛГЕБРЫ С НУЛЕВЫМ УМНОЖЕНИЕМ

Построен базис универсальной мультипликативной обертывающей алгебры левосимметричной алгебры. Описаны аффинные автоморфизмы универсальной мультипликативной обертывающей алгебры двумерной левосимметричной алгебры с нулевым умножением.

***Ключевые слова**: левосимметричная алгебра, универсальная мультипликативная обертывающая алгебра, автоморфизм.*