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A. A. SARSENBI [1,2]*, E. MUSSIREPOVA[2].

[1]Academician A. Kuatbekov University of Peoples' Friendship, Shymkent, Kazakhstan
[2]M. Auezov South Kazakhstan University, Shymkent, Kazakhstan

GREEN'S FUNCTION OF A BOUNDARY VALUE PROBLEM FOR A SECOND-ORDER DIFFERENTIAL EQUATION WITH INVOLUTION

A boundary value problem for a second-order differential equation with involution in the second derivative is considered. The existence of linearly independent solutions of the studied equation is established. The definition of the Green's function is given. The existence of eigenvalue problems for eigenvalue problems is proved. The Green's function is constructed in the case of a Dirichlet-type problem.

***Key words**: ODE with involution; Green's function; eigenvalue*

Introduction. In this paper, we study some spectral properties of boundary value problems for a second-order differential equation with involution in the second derivative. The obtained results are formulated for the case of simple boundary value problems, although they are valid in the case of general boundary conditions. Let us consider the equation:

$$Ly \equiv -y''(x) + \alpha y''(-x) + q(x)y(x) = 0,\ -1 < x < 1, \tag{1}$$

where $-1 < \alpha < 1$, $q(x) \in C[-1,1]$. Equation (1) contains an involution transformation of the following type $(Sy)(x) = y(-x)$ for any function $y(x) \in L_2(-1,1)$.

In the scientific literature there are many works devoted to various problems of the theory of differential equations with involution. Bibliography on this subject can be found in monographs by D. Przeworska-Rolewicz [1], J. Wiener [2], Alberto Cabada and F. Adrian F. Tojo [3]. Despite this, spectral aspects for differential equations with involution are still poorly studied. Spectral problems for the first-order equation with involution were first studied in the work of T.Sh. Kalmenov [4]. The problems for the first-order equation with involution were also studied in the works of A.P. Khromov [5,6] and A.G. Baskakov [7]. Spectral problems for the second-order differential equations with involution are the subject of works [8-14].

In [12], [13], [15], the questions of the basis property of eigenfunctions of the boundary value problems for the second-order differential equations with involution were studied using the Green's function for the studied problems. However, the properties of the Green's function of boundary value problems for the second-order differential equations with involution are not fully formulated, as was done in the case of boundary value problems for ordinary differential equations [16]-[17].

Green's function of the boundary value problem. Consider a second-order homogeneous differential equation with involution in the second derivative

$$y''(x) + \alpha y''(-x) + q(x)y(x) = 0 \tag{1}$$

* E-mail корреспондирующего автора: abdisalam@mail.ru

with coefficient $q(x)\in C(-1,1)$. As in the case of ordinary differential equations, equation (1) satisfies the following

Theorem 1. The homogeneous differential equation with involution (1) has two linearly independent solutions.

Proof. Note that equation (1) has a non-local character, as it connects the values of the unknown function at two points. It is known from the theory of ordinary differential equations that the existence theorems are of a local nature. Equation (1) becomes local in a small neighborhood of the point $x = 0$. Therefore, the existence and uniqueness theorem for the solution of the Cauchy problem must hold for it. This fact was proved in the work of M.A. Sadybekov and his followers [18].

Consider for equation (1) the Cauchy problem with the initial data $y(0)=a_{11}, y'(0)=a_{21},$. $y(0)=a_{12}$, $y'(0)=a_{22}$. We denote the corresponding solutions by $y_1(x), y_2(x)$. As $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$ is the Vronsky determinant of these solutions, then these solutions are linearly independent. The theorem has been proved.

It is not difficult to check that the general solution of Eq. (1) has the form

$$y(x)=c_1y_1(x)+c_2y_2(x) \ . \tag{2}$$

For simplicity, consider equation (1) with antiperiodic boundary conditions

$$y(-1)=-y(1), y'(-1)=-y'(1) \ . \tag{3}$$

The solution to the boundary value problem (1), (3) is a twice continuously differentiable function $y(x)\in C^1[-1,1]$, satisfying equation (1) and boundary conditions (2). It may turn out that the boundary value problem (1), (3) does not have a nonzero solution. Then there may exist such a function $G(x,t)$ that the following conditions are satisfied:

1) the function $G(x,t)$ is continuous for $x,t\in[-1,1]$;

2) the function $G(x,t)$ has a discontinuous first derivative with respect to x for $x\neq\mp t$ for fixed t and

$$G'_x(-t-0,t)-G'_x(-t+0,t)=\alpha\left(1-\alpha^2\right)^{-1},$$

$$G'_x(t-0,t)-G'_x(t+0,t)=\left(1-\alpha^2\right)^{-1},$$

3) the function $G(x,t)$ has a second derivative with respect to x for $x\in[-1,-t)\cup(-t,t)\cup(t,1]$, satisfies equation (1) and boundary conditions (3).

The function satisfying these three conditions will be called the Green's function of the boundary value problem (1), (3). The following theorem is valid.

Theorem 2. The boundary value problem (1), (3) has a unique Green's function if it has only a trivial solution.

Proof. According to Theorem 1, the boundary value problem has a general solution (2). By definition, the Green's function satisfies equation (1) on each of the sets $[-1,-t)$, $(-t,t)$, $(t,1]$. In the interval $[-1,-t)$ it is written as

$$G(x,t)=a_1y_1(x)+a_2y_2(x),\ -1\le x<-t\ .$$

In the intervals $(-t,t)$, $(t,1]$ it has the forms

$$G(x,t)=b_1y_1(x)+b_2y_2(x),\ -t<x<t\ ,$$

$$G(x,t)=c_1y_1(x)+c_2y_2(x),\ t<x\le 1\ .$$

According to condition 1) in the definition of the Green's function we will have the following equalities

$$\begin{aligned} &a_1y_1(-t)+a_2y_2(-t)-b_1y_1(-t)-b_2y_2(-t)=0,\\ &b_1y_1(t)+b_2y_2(t)-c_1y_1(t)-c_2y_2(t)=0. \end{aligned} \tag{4}$$

According to condition 2) in the definition of the Green's function we get

$$\begin{aligned} &a_1y_1'(-t)+a_2y_2'(-t)-b_1y_1'(-t)-b_2y_2'(-t)=\alpha\left(1-\alpha^2\right)^{-1},\\ &b_1y_1'(t)+b_2y_2'(t)-c_1y_1'(t)-c_2y_2'(t)=-\left(1-\alpha^2\right)^{-1}. \end{aligned} \tag{5}$$

Equations (4), (5) can be rewritten as

$$\begin{aligned} &\gamma_{11}y_1(-t)+\gamma_{12}y_2(-t)=0,\\ &\gamma_{11}y_1'(-t)+\gamma_{12}y_2'(-t)=\alpha\left(1-\alpha^2\right)^{-1}, \end{aligned} \tag{6}$$

$$\begin{aligned} &\gamma_{21}y_1(t)+\gamma_{22}y_2(t)=0,\\ &\gamma_{21}y_1'(t)+\gamma_{22}y_2'(t)=-\left(1-\alpha^2\right)^{-1}, \end{aligned} \tag{7}$$

where

$$\gamma_{11}=a_1-b_1, \gamma_{12}=a_2-b_2\ , \gamma_{21}=b_1-c_1, \gamma_{22}=b_2-c_2. \tag{8}$$

Equalities (6), (7), respectively, imply the uniqueness of the pair of numbers λ_{11}, λ_{12} and λ_{21}, λ_{22}. To prove the uniqueness, we use the boundary conditions (3)

$$\begin{aligned} &a_1y_1(-1)+a_2y_2(-1)+c_1y_1(1)+c_2y_2(1)=0,\\ &a_1y_1'(-1)+a_2y_2'(-1)+c_1y_1'(1)+c_2y_2'(1)=0. \end{aligned} \tag{9}$$

Then from (8) we get

$$a_1=c_1+\gamma_{11}+\gamma_{21},\quad a_2=c_2+\gamma_{12}+\gamma_{22}.$$

Then equation (9) can be rewritten as

$$c_1\left(y_1(-1)+y_1(1)\right)+c_2\left(y_2(-1)+y_2(1)\right)=-(\gamma_{11}+\gamma_{21})y_1(-1)-(\gamma_{12}+\gamma_{22})y_2(-1),$$
$$c_1\left(y_1'(-1)+y_1'(1)\right)+c_2\left(y_2'(-1)+y_2'(1)\right)=-(\gamma_{11}+\gamma_{21})y_1'(-1)-(\gamma_{12}+\gamma_{22})y_2'(-1).$$

The boundary value problem (1), (3) has no nonzero solutions. That's why $\left(y_1(-1)+y_1(1)\right)\neq 0$, $\left(y_2(-1)+y_2(1)\right)\neq 0$, $\left(y_1'(-1)+y_1'(1)\right)\neq 0$, $\left(y_2'(-1)+y_2'(1)\right)\neq 0$. Due to the linear independence of the boundary forms, the determinant of the last system of equations is nonzero. Therefore, the quantities c_1, c_2 are determined in a unique way. Then from equalities (8) the quantities a_1, a_2, b_1, b_2 are uniquely determined. The theorem is proved.

Note that the theorem is true for any boundary conditions, provided that the linear forms are linearly independent

$$U_1(y)=a_{11}y'(-1)+a_{12}y'(1)+a_{13}y(-1)+a_{14}y(1),$$
$$U_2(y)=a_{21}y'(-1)+a_{22}y'(1)+a_{23}y(-1)+a_{24}y(1).$$

Theorem 3. If the boundary value problem (1), (3) has no nontrivial solutions, then for any continuous function $f(x)\in C[-1,1]$ the non-homogeneous problem

$$y''(x)+\alpha y''(-x)+q(x)y(x)=f(x) \tag{10}$$

with the same boundary conditions has a unique solution written as

$$y(x)=\int_{-1}^{1}G(x,t)f(t)dt\ , \tag{11}$$

where $G(x,t)$ is the Green's function of the homogeneous boundary value problem.

Proof. After differentiating the function (11) two times, we obtain the relation

$$\begin{aligned}y''(x)=&\int_{-1}^{-x}G_x''(x,t)f(t)dt+\int_{-x}^{x}G_x''(x,t)f(t)dt+\int_{x}^{1}G_x''(x,t)f(t)dt-\\&-\left[G_x'(x,t)\big|_{t=-x-0}-G_x'(x,t)\big|_{t=-x+0}\right]f(-x)-\left[G_x'(x,t)\big|_{t=x+0}-G_x'(x,t)\big|_{t=x-0}\right]f(x),\end{aligned} \tag{12}$$

After replacing by we get

$$\begin{aligned}y''(-x)=&\int_{-1}^{-x}G_x''(-x,t)f(t)dt+\int_{-x}^{x}G_x''(-x,t)f(t)dt+\int_{x}^{1}G_x''(-x,t)f(t)dt+\\&+\left[G(-x,t)'_x\Big|_{t=-x+0}-G(-x,t)'_x\Big|_{t=-x-0}\right]f(-x)+\\&+\left[G(-x,t)'_x\Big|_{t=x+0}-G(-x,t)'_x\Big|_{t=x-0}\right]f(x).\end{aligned} \tag{13}$$

Note that

$$\begin{aligned} &G(-x,t)'_x\Big|_{t=-x-0} - G(-x,t)'_x\Big|_{t=-x+0} = \left(1-\alpha^2\right)^{-1}, \\ &G(-x,t)'_x\Big|_{t=x+0} - G(-x,t)'_x\Big|_{t=x-0} = \alpha\left(1-\alpha^2\right)^{-1}. \end{aligned} \tag{14}$$

$$\begin{aligned} &G'_x(-t-0,t) - G'_x(-t+0,t) = \alpha\left(1-\alpha^2\right)^{-1}, \\ &G'_x(t-0,t) - G'_x(t+0,t) = \left(1-\alpha^2\right)^{-1}. \end{aligned}$$

Substituting relations (12), (13), (14) into equation (10), we obtain the equality

$$\frac{\alpha}{\sqrt{1-\alpha^2}} f(-x) + \frac{1}{\sqrt{1-\alpha^2}} f(x) - \frac{\alpha}{\sqrt{1-\alpha^2}} f(-x) - \frac{\alpha^2}{\sqrt{1-\alpha^2}} f(x) = f(x).$$

Therefore, function (11) satisfies equation (1). The fulfillment of the boundary conditions is verified directly, since the Green's function, by definition, satisfies the boundary conditions. The theorem has been proved.

The proved theorem is also valid for the boundary conditions

$$\begin{aligned} &U_1(y) = a_{11}y'(-1) + a_{12}y'(1) + a_{13}y(-1) + a_{14}y(1) = 0, \\ &U_2(y) = a_{21}y'(-1) + a_{22}y'(1) + a_{23}y(-1) + a_{24}y(1) = 0. \end{aligned} \tag{15}$$

Existence of eigenvalues of boundary value problems for a second-order differential equation with involution. In the general theory of differential equations, eigenvalue problems occupy an important place. Consider the boundary value problem

$$y''(x) + \alpha y''(-x) + q(x)y(x) = \lambda y(x) \tag{16}$$

with boundary conditions (3). The question is raised about the existence of eigenvalues of the boundary value problem (16), (3).

Let the boundary value problem (1), (3) have only a zero solution. Then, by Theorem 3, the boundary value problem (16), (3) is equivalent to the integral equation

$$y(x) = \lambda\int_{-1}^{1} G(x,t)y(t)dt. \tag{17}$$

Due to continuity of the Green's function $G(x,t)$, the theory of Fredholm integral equations can be applied to the integral equation (17), according to which the integral equation (17) has infinite number of eigenvalues with a single limit point at infinity. Therefore, the following assertion is valid.

Assertion. If the boundary value problem (1), (3) does not have a nontrivial solution, then the boundary value problem (16), (3) has an infinite number of eigenvalues with a single limit point at infinity.

*Exampl*e. Consider a Dirichlet-type problem for the following second-order differential equation with an involution

$$-y''(x)+\alpha y''(-x)=0, -1<x<1,\ -1<\alpha<1, \tag{18}$$

$$y(-1)=0, y(1)=0\ , \tag{19}$$

It is not difficult to check that the boundary value problem (18), (19) has only a zero solution. Then it has a Green's function. By direct calculation, one can verify that the function

$$G(x,t)=\frac{1}{2(1-\alpha)}-\frac{xt}{2(1+\alpha)}+\frac{1}{2}\begin{cases}\frac{t}{1-\alpha}-\frac{x}{1+\alpha}, t\le -x;\\ -\frac{x}{1-\alpha}+\frac{t}{1+\alpha}, -x\le t\le x;\\ -\frac{t}{1-\alpha}+\frac{x}{1+\alpha}, t\le x.\end{cases}$$

is the Green's function of the boundary value problem (18), (19). To do this, it suffices to show that the function $y(x)=\int_{-1}^{1} G(x,t)\, f(t)\,dt$ satisfies the equation $-y''(x)+\alpha y''(-x)=f(x)$ and boundary conditions (19) for any continuous function $f(x)$.

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Ә. Ә. СӘРСЕНБІ[1,2], Э. Б. МҮСІРЕПОВА[2]

[1]Академик А. Куатбеков атындағы халықтар достығы университеті, Шымкент, Қазақстан
[2]М.Әуезов атындағы ОҚУ, Шымкент, Қазақстан

ИНВОЛЮЦИЯСЫ БАР ЕКІНШІ РЕТТІ ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУ ҮШІН ШЕКАРАЛЫҚ ЕСЕПТІҢ ГРИН ФУНКЦИЯСЫ

Екінші ретті туындысында инволюциясы бар дифференциалды теңдеу үшін шекаралық есеп қарастырылады. Зерттелетін теңдеудің сызықтық тәуелсіз шешімдерінің бар екендігі анықталды. Грин функциясының анықтамасы берілген. Шеттік есептің меншікті мәндері бар екендігі дәлелденді. Дирихле шеттік есебінің Грин функциясы айқын түрде жазылды.

***Түйін сөздер**: инволюциясы бар дифференциалды теңдеулер, Грин фукнциясы, меншікті мәндер.*

Ә. Ә. СӘРСЕНБІ[1,2], Э. Б. МҮСІРЕПОВА[2]

[1]Университет дружбы народов им. академика А. Куатбекова, Шымкент, Казахстан
[2]ЮКУ им. М. Ауэзова, Шымкент, Казахстан

ФУНКЦИЯ ГРИНА КРАЕВОЙ ЗАДАЧИ ДЛЯ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ ВТОРОГО ПОРЯДКА С ИНВОЛЮЦИЕЙ

Рассматривается краевая задача для дифференциального уравнения второго порядка с инволюцией во второй производной. Установлено существование линейно независимых решений изучаемого уравнения. Дано определение функции Грина. Доказано существование собственных значений задач на собственные значения. Построена функция Грина в случае задачи типа Дирихле.

Ключевые слова*: ОДУ с инволюцией; функция Грина; собственное значение.*