UDC 517.957
https://doi.org/10.47533/2023.1606-146X.12

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GENERALIZED (G'/G) - EXPANSION METHOD FOR THE LOADED SHALLOW WATER WAVE EQUATION

This article is devoted to finding solutions for the traveling wave of the loaded wave equation in shallow water. One of the approaches to finding solutions by the expansion method (G'/G) is given, which is one of the most effective ways to obtain solutions. When parameters are taken as special values, solitary waves are also derived from traveling waves. Solutions for the traveling wave are expressed by hyperbolic and trigonometric functions. This method is easy to implement using well-known software packages that allow solving complex nonlinear evolutionary equations of mathematical physics.

***Key words**: expansion method, evolution equations, continuous function, loaded equation, shallow water wave equation.*

Introduction. The work [1] presents the use of the – expansion method (G'/G) for integrating nonlinear evolutionary equations. The expansion method (G'/G) is also often used in finding solutions to nonlinear evolutionary traveling wave equations [2-12].

There are well-known works [13-14], where this method was effectively applied to solutions of the loaded Korteweg-de Vries equation (KdV) and the modified Korteweg-de Vries equation.

This research shows one of the options for solving a loaded wave equation using a generalized extension method.

Let us turn to the loaded equation of a traveling wave in shallow water

$$q_{xxxt} + \alpha q_x q_{xt} + \beta q_t q_{xx} - q_{xt} - \gamma q_{xx} + f(t)q(0,t)q_{xx} = 0 , \quad (1)$$

where $q(x,t)$ – unknown function, $x \in R$, $t \geq 0$, $f(t)$ – given continuous function

Algorithm of the generalized (G'/G) expansion method. Let us consider the following non-linear equation

$$F(q, q_t, q_x, q_{tt}, q_{xx}, q_{xt}, ...) = 0 , \quad (2)$$

here $q = q(x,t)$ unknown function of independent variables *x* and *t*. *F* is a polynomial of *q* and its partial derivatives of higher- order and non-linear terms. By stages, the main steps of the expansion method are as follows [2]:

Step 1. We are looking for a business trip form:

$$q(x,t) = q(\xi) , \xi = x - \Omega(t) , \quad (3)$$

where $\Omega(t)$ continuous parameter is a function of a variable *t*. Now we transform equation (2) to the following nonlinear ODE:

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$$P(q,q',q'',q''',...)=0\ , \qquad (4)$$

where *P* - expresses a polynomial depending on $q(\xi)$ and $q'=dq(\xi)/d\xi$, $q''=d^2q(\xi)/d\xi^2$.

Step 2. We assume that the solution of equation (4) can be represented as

$$q(\xi)=\sum_{j=0}^{m}a_j\left(\frac{G'}{G}\right)^j\ , \qquad (5)$$

where $G=G(\xi)$ satisfies the following second- order ordinary differential equation

$$G''+\lambda G'+\mu G=0\ , \qquad (6)$$

here $G'=dG(\xi)/d\xi$, $G''=d^2G(\xi)/d\xi^2$ and λ, μ, $a_j(j=1,2,...,m)$ are constants whose values will be determined later, assuming $a_m \neq 0$.

Step 3. Such an integer is determined *m*, to balance the higher-order nonlinear terms and higher-order partial derivatives from (4).

Step 4. Using (5) and (6) we transform (4) and group all terms with the same order $\left(\frac{G'(\xi)}{G(\xi)}\right)$. Then the left side of expression (4) is transformed into a polynomial $\left(\frac{G'(\xi)}{G(\xi)}\right)$. Now, equating to zero each coefficient of this polynomial, we obtain overdetermined partial differential equations for $a_j(j=1,2,...,m)$ and ξ.

Step 5. Substituting the solutions of equation (6) into (5), as well as the values a_j and ξ, we will come to an exact solution of the equation (2).

Exact solutions of a loaded nonlinear equation. This section provides an exact solution to a loaded nonlinear equation using (G'/G) expansion method. To this end, we perform the above steps for equation (1). Using a traveling wave variable

$$q(x,t)=q(\xi)\ ,\ \xi=x-\Omega(t)\ , \qquad (7)$$

Let us transform equation (1) into an ordinary differential equation with respect to $q = q(\zeta)$

$$-\Omega_t'(t)q^{IV}-\alpha\Omega_t'(t)q'q''-\beta\Omega_t'(t)q'q''+\Omega_t'(t)q''-\gamma q''+f(t)q(0,t)q''=0\ , \qquad (8)$$

integrating it over ξ we get

$$C-\Omega_t'(t)q'''-\frac{1}{2}\Omega_t'(t)(\alpha+\beta)(q')^2+(\Omega_t'(t)-\gamma)q'+f(t)q(0,t)q'=0\ , \qquad (9)$$

where *C*- integration constant, the value of which can be determined in the future. The solution of equation (9) in the form of a polynomial can be expressed as

$$q(\xi)=\sum_{j=0}^{m}a_j\left(\frac{G'}{G}\right)^j\ , \qquad (10)$$

where $G = G(\xi)$ satisfies a second-order ordinary differential equation of the form

$$G'' + \lambda G' + \mu G = 0\ . \tag{11}$$

Using (10) and (11), q'^2 and q''' can be easily obtained

$$q'^2(\xi) = m^2 a_m^2 \left(\frac{G'}{G}\right)^{2(m+1)} + \ldots \tag{12}$$

$$q'''(\xi) = m(m+1)(m+2)a_m \left(\frac{G'}{G}\right)^{m+3} + \ldots\ . \tag{13}$$

When the condition $m = 1$ and taking into account a homogeneous balance between $q'^2(\xi)$ and $q'''(\xi)$ in equation (9), based on (12) and (13), the expression for q will take as follows

$$q(\xi) = a_1\left(\frac{G'}{G}\right) + a_0\ , \tag{14}$$

then the following relation holds

$$\begin{gathered} q'^2(\xi) = \left(a_1\left(\frac{G'}{G}\right)^2 + \lambda a_1\left(\frac{G'}{G}\right) + a_1\mu\right)^2 = a_1^2\left(\frac{G'}{G}\right)^4 + \lambda^2 a_1^2\left(\frac{G'}{G}\right)^2 + \\ + a_1^2\mu^2 + 2a_1^2\lambda\left(\frac{G'}{G}\right)^3 + 2a_1^2\mu\left(\frac{G'}{G}\right)^2 + 2a_1^2\lambda\mu\left(\frac{G'}{G}\right). \end{gathered} \tag{15}$$

After simple transformations based on (14) and (11), we obtain an expression for $q'''(\xi)$

$$\begin{gathered} q'''(\xi) = -6a_1\left(\frac{G'}{G}\right)^4 - 12a_1\lambda\left(\frac{G'}{G}\right)^3 - (7a_1\lambda^3 + 8a_1\mu)\left(\frac{G'}{G}\right)^2 - \\ -(a_1\lambda^3 + 8a_1\lambda\mu)\left(\frac{G'}{G}\right) - (a_1\lambda^2\mu + 2a_1\mu^2)\ . \end{gathered} \tag{16}$$

Further, after substituting relations (15)-(16) into equation (9) and grouping all terms with the same degree (G'/G), on the left side of equation (9) we obtain a polynomial

$$\begin{gathered} (6a_1\Omega_t'(t) - \frac{1}{2}\Omega_t'(t)(\alpha+\beta)a_1^2)\left(\frac{G'}{G}\right)^4 + (12a_1\lambda\Omega_t'(t) - \Omega_t'(t)(\alpha+\beta)a_1^2\lambda)\left(\frac{G'}{G}\right)^3 + \\ +(7a_1\lambda^2\Omega_t'(t) + 8a_1\mu\Omega_t'(t) - \frac{1}{2}\Omega_t'(t)(\alpha+\beta)(a_1^2\lambda^2 + 2a_1^2\mu) - (\Omega_t'(t) - \gamma) - f(t)q(0,t)a_1)\left(\frac{G'}{G}\right)^2 + \\ +(a_1\lambda^3\Omega_t'(t) + 8a_1\mu\lambda\Omega_t'(t) - \Omega_t'(t)(\alpha+\beta)a_1^2\lambda\mu - a_1\lambda(\Omega_t'(t) - \gamma) - a_1\lambda f(t)q(0,t))\left(\frac{G'}{G}\right) + \end{gathered} \tag{17}$$

$$+(C+\Omega_t'(t)(\lambda^2 a_1\mu+2a_1\mu^2)-\frac{1}{2}\Omega_t'(t)(\alpha+\beta)a_1^2\mu^2-a_1\mu(\Omega_t'(t)-\gamma)-a_1\mu f(t)q(0,t))\left(\frac{G'}{G}\right)^0=0$$

If each coefficient of expression (17) is equated to zero, we obtain a system of homogeneous equations for a_1, $\Omega(t)$ and C:

$$\left(\frac{G'}{G}\right)^4: 6a_1\Omega_t'(t)-\frac{1}{2}\Omega_t'(t)(\alpha+\beta)a_1^2=0\ ,$$

$$\left(\frac{G'}{G}\right)^3: 12a_1\lambda\Omega_t'(t)-\Omega_t'(t)(\alpha+\beta)a_1^2\lambda=0\ ,$$

$$\left(\frac{G'}{G}\right)^2: 7a_1\lambda^2\Omega_t'(t)+8a_1\mu\Omega_t'(t)-\frac{1}{2}\Omega_t'(t)(\alpha+\beta)(a_1^2\lambda^2+2a_1^2\mu)-(\Omega_t'(t)-\gamma)a_1-f(t)q(0,t)a_1=0\ ,$$

$$\left(\frac{G'}{G}\right)^1: a_1\lambda^3\Omega_t'(t)+8a_1\mu\lambda\Omega_t'(t)-\Omega_t'(t)(\alpha+\beta)a_1^2\lambda\mu-a_1\lambda(\Omega_t'(t)-\gamma)-a_1\lambda f(t)q(0,t)=0\ ,$$

$$\left(\frac{G'}{G}\right)^0: C+\Omega_t'(t)(\lambda^2 a_1\mu+2a_1\mu^2)-\frac{1}{2}\Omega_t'(t)(\alpha+\beta)a_1^2\mu^2-a_1\mu(\Omega_t'(t)-\gamma)-a_1\mu f(t)q(0,t)=0\ .$$

The solution of the system will be

$$a_1=\frac{12}{\alpha+\beta}\ ,\ C=0\ ,$$

$$\Omega(t)=-\frac{\gamma}{\lambda^2-4\mu-1}t+\frac{1}{\lambda^2-4\mu-1}\int_0^t f(\tau)q(0,\tau)d\tau+\Omega^0\ , \tag{18}$$

λ, μ and Ω^0 are arbitrary constants.

Using the relation (18), the expression (14) can be represented as an equality

$$q(\xi)=\frac{12}{\alpha+\beta}\left(\frac{G'}{G}\right)+a_0\ , \tag{19}$$

where

$$\xi=x+\frac{\gamma}{\lambda^2-4\mu-1}t-\frac{1}{\lambda^2-4\mu-1}\int_0^t f(\tau)q(0,\tau)d\tau-\Omega^0\ .$$

Function (19) will express the solution of equation (9) in the case when the integration constant in equation (9) corresponds to similar conditions (18). When substituting the general solution of equation (11) into (19), three types of solutions of the loaded traveling wave equation (1) for shallow water are obtained when $\lambda^2-4\mu>0$,

$$q(\xi) = a_0 + 6\frac{\left[(c_1\sqrt{\lambda^2-4\mu} - c_2\lambda) + (c_2\sqrt{\lambda^2-4\mu} - c_1\lambda)\tanh\sqrt{\frac{\lambda^2-4\mu}{4}}\xi\right]}{(\alpha+\beta)(c_2 + c_1\tanh\sqrt{\frac{\lambda^2-4\mu}{4}}\xi)}, \qquad (20)$$

where

$$\xi = x + \frac{\gamma}{\lambda^2-4\mu-1}t - \frac{1}{\lambda^2-4\mu-1}\int_0^t f(\tau)q(0,\tau)d\tau - \Omega^0 ,$$

c_1, c_2 and Ω^0 are arbitrary constants. It is clear that finding a function $q(0, t)$ will not cause difficulties based on expression (20).

For example, let the function $f(t)$ be given in the form

$$f(t) = \left(\gamma t - (\lambda^2-4\mu-1)\sum_{j=1}^{n} j\alpha_j t^{j-1}\right)\times$$

$$\times\frac{(\alpha+\beta)\tanh\sqrt{\frac{\lambda^2-4\mu}{4}}\sum_{j=1}^{n}\alpha_j t^j}{a_0(\alpha+\beta)\tanh\sqrt{\frac{\lambda^2-4\mu}{4}}\sum_{j=1}^{n}\alpha_j t^j - 6(\sqrt{\lambda^2-4\mu} + \lambda\tanh\sqrt{\frac{\lambda^2-4\mu}{4}}\sum_{j=1}^{n}\alpha_j t^j)},$$

where $\alpha_j\ (j = 1, 2, ..., n)$, if $c_1 \neq 0$, $c_2 = 0$ and $\lambda^2 - 4\mu > 0$, that $q(x,t)$ can be expressed

$$q(x,t) = \frac{a_0(\alpha+\beta)\tanh\sqrt{\frac{\lambda^2-4\mu}{4}}(x - \sum_{j=1}^{n}\alpha_j t^j) + 6(\sqrt{\lambda^2-4\mu} - \lambda\tanh\sqrt{\frac{\lambda^2-4\mu}{4}}(x - \sum_{j=1}^{n}\alpha_j t^j))}{(\alpha+\beta)\tanh\sqrt{\frac{\lambda^2-4\mu}{4}}(x - \sum_{j=1}^{n}\alpha_j t^j)}. \qquad (21)$$

Then the last relation (21) will be the solution of the equation of a loaded wave in shallow water

$$q_{xxxt} + \alpha q_x q_{xt} + \beta q_t q_{xx} - q_{xt} - \gamma q_{xx} - \left((\lambda^2-4\mu-1)\sum_{j=1}^{n} j\alpha_j t^{j-1} - \gamma t\right)q_{xx} = 0 . \qquad (22)$$

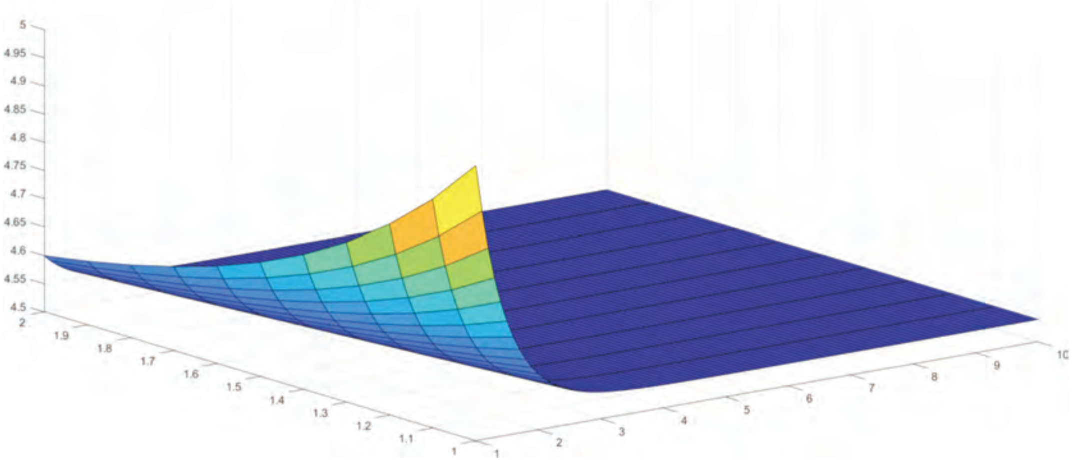

Figure 1 – Solution (21) of the loaded wave equation in shallow water (22) for
$\lambda = 2\sqrt{2},\ \mu = 1,\ a_0 = 1,\ \alpha = 1,\ \beta = 1,\ n = 1,\ \alpha_1 = -1,\ t \in [1,2],\ x \in [1,10]$

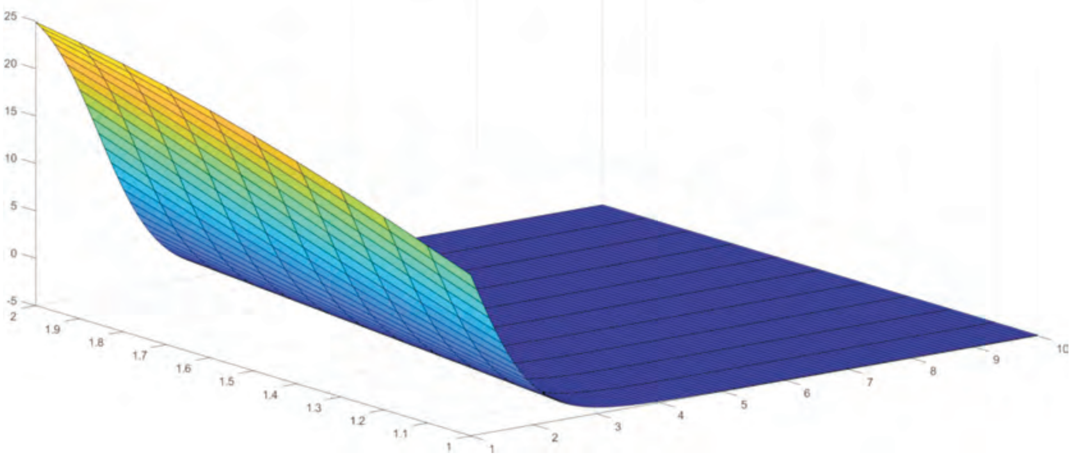

Figure 2 – Solution (21) of the loaded wave equation in shallow water (22) for
$\lambda = 2\sqrt{2},\ \mu = 1,\ a_0 = 1,\ \alpha = -1,\ \beta = 1,\ n = 1,\ \alpha_1 = 1,\ t \in [1,2],\ x \in [1,10]$

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ЖАЛПЫЛАНҒАН ТАЯЗ СУДЫҢ ЖҮКТЕЛГЕН ТОЛҚЫНДЫҚ ТЕҢДЕУІНЕ АРНАЛҒАН (G'/G) КЕҢЕЙТУ ӘДІСІ

Бұл мақала таяз суларда жүктелген толқындық теңдеудің шешімдерін табуға арналған. Шешімдерді алудың ең тиімді әдістерінің бірі болып табылатын (G'/G) кеңейту әдісімен шешімдерді іздеу тәсілдерінің бірі келтірілген. Параметрлер арнайы мәндер ретінде алынған кезде, жалғыз толқындар да қозғалатын толқындардан шығарылады. Жүгіру толқынының шешімдері гиперболалық және тригонометриялық функциялармен көрсетілген. Бұл әдіс математикалық *физиканың күрделі сызықтық емес эволюциялық теңдеулерін шешуге мүмкіндік беретін белгілі бағдарламалық пакеттерді қолдану арқылы оңай жүзеге асырылады.*

***Түйін сөздер**: кеңейту әдісі, эволюциялық теңдеулер, үздіксіз функция, жүктелген теңдеу, таяз судағы толқындық теңдеу.*

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ОБОБЩЕННЫЙ (G'/G) МЕТОД РАСШИРЕНИЯ ДЛЯ НАГРУЖЕННОГО ВОЛНОВОГО УРАВНЕНИЯ МЕЛКОВОДЬЯ

Данная статья посвящена нахождению решений нагруженного волнового уравнения бегущей волны на мелководье. Приводится один из подходов поиска решений (G'/G) методом расширения, который является одним из наиболее действенных способов получения решений. Когда параметры берутся в качестве специальных значений, уединенные волны также выводятся из бегущих волн. Решения для бегущей волны выражаются гиперболическими и тригонометрическими функциями. Этот метод легко реализовать с использованием известных программных пакетов, которые позволяют решать сложные нелинейные эволюционные уравнения математической физики.

***Ключевые слова**: метод расширения, эволюционные уравнения, непрерывная функция, нагруженное уравнение, волновое уравнение на мелководье.*