UDC: 511.11
https://doi.org/10.47533/2023.1606-146X.45

ALIKHAN ZHARBOLOV

National Physics and Mathematics School, Almaty, Kazakhstan.
E-mail: zharbolov@gmail.com

THE RANK OF THE NUMBERS OF THE A-FUNCTION

The famous Indian mathematician D. Kaprekar came up with a method for obtaining new numbers $K(n) = n + S(n)$ where $S(n)$ is the sum of the digits of a number n in decimal notation. Following D. Kaprekar, the paper presents a new way to obtain numbers

$$A(n) = \frac{1}{9}\left(n - S(n)\right).$$

It is proved that the function $A(n)$ is non-decreasing. Similarly to the function $K(n)$, the concepts of a-generated and a-self-generated numbers are defined. The rank of numbers is investigated and it is proved that the rank function is non-decreasing. The rank formula for the numbers $n = 10^S$
Keywords: *D.K.Kaprekar, self-corrected numbers, A – function.*

1. Introduction. The Indian mathematician D.R.Kaprekar discovered several remarkable classes of natural numbers, such as the Kaprekar constant[1], Kaprekar numbers[2], Harshad numbers, Demlo numbers[3].

Another outstanding discovery of D.Kaprekar is the class of self-generated numbers. It is described by the famous popularizer of science Martin Gardner in his book "Time Travel"[4]. Let's choose any natural number n and add to it the sum of its digits $S(n)$. The resulting number $K(n) = n + S(n)$ is called the generated number, and the original number is called its generator. For example, if we choose the number 65, then the number generated by it is 65 + 6 + 5 = 76.

A generated number can have more than one generator. The smallest number with two generators is 101, and its generators are the numbers 91 and 100. A self-generated number is a number that does not have a generator. A 1974 article in American Mathematical Monthly[5] argued that there are infinitely many self-generated numbers, but they are much rarer than generated numbers. Within the first hundred, there are only 13 self-generated numbers: 1, 3, 5, 7, 9, 20, 31, 42, 53, 64, 75, 86, 97. Among the powers of the number 10 million, i.e. 10 is a self-generated number. The next power of a million to the power of ten, which is a self-generated number, is 10^{16}.

These discoveries of D.Kaprekar interested many mathematicians, and in different countries of the world there were many scientific articles, scientific projects in mathematics, software products in which various new properties of the «Kaprekar constant» and sets of self-generated numbers and generated numbers were investigated. These works were carried out not only by scientists, but also by many high school students. There are also many Olympiad problems in mathematics related to n, $S(n)$. Therefore, I believe that these discoveries of D.Kaprekar greatly raised the interest of young people in mathematics.

In this paper, a new method of obtaining integers is introduced. The difference between a number and the sum of its numbers $n – S(n)$ is always divisible by 9. Therefore, the new function for obtaining integers is defined as follows: $A(n) = \frac{1}{9}\big(n - S(n)\big)$. It is proved that the function $A(n)$ is non-decreasing. Similar to Kaprekar, for the case of a function $A(n)$, classes of a–generated and a–self-generated numbers are defined. The rank of the numbers of the function $A(n)$ is investigated and it is shown that the rank function is a non-decreasing function. The smallest and largest number of a given rank s are found, where $1 \le s \le 15$. We derived the rank formula for the numbers $n = 10^s$ where $s \in N$.

2. Formula *A*(*n*). Let and $N = \{1,2,3,...\}$ be a set of natural numbers. Let us denote $N_0 = N \cup \{0\}$ through - the set of non-negative integers. Throughout this work, the number system is decimal. Let $n \in N$ and

$$n = \alpha_0 + \alpha_1 \cdot 10 + \cdots + \alpha_k \cdot 10^k = \sum_{i=1}^{k} \alpha_i \cdot 10^i, \quad \text{where} \quad \alpha_k \neq 0.$$

Let's denote the sum of the digits of the number n by

$$S(n) = \alpha_0 + \alpha_1 + \cdots + \alpha_k = \sum_{i=0}^{k} \alpha_i.$$

The difference between the number and the sum of its digits $n - S(n)$ is a multiple of 9. Therefore, in this paper, a new function for obtaining integers is introduced.

$$A(n) = \frac{1}{9}\big(n - S(n)\big), \quad \text{where} \quad n \in N, \quad A(n) \in N_0 \tag{1}$$

Definition. Let $A(n) = m$, where $m, n \in N_0$. The number is called a – generated number, and the number n is called its a – generator. A-Generated number can have multiple generators. If a number p does not have a-generators, then it is called a-self number. Examples of a are self-generated numbers: 10, 20, 32, 109, 110, 1108, 1109, 1110 and others.

Definition. Let $m, n \in N, \quad m < n,$

$$m = \sum_{j=0}^{k} \beta_j \cdot 10^j, \qquad n = \sum_{i=0}^{k} \alpha_i \cdot 10^i, \quad \alpha_k \neq 0.$$

Let $\alpha_i - \beta_i = 0$ if $i = l + 1, l + 2, ..., k$, but $\alpha_l - \beta_l \ge 1$. Then the number l is called the «difference» of the powers of the numbers n and m and is denoted by

$$l(n, m) = 1$$

Theorem 1. $A(n) - A(m) \ge l(n, m)$.

Proof. $A(n) - A(m) = \frac{1}{9}\big(n - S(n)\big) - \frac{1}{9}\big(m - S(m)\big) = \frac{1}{9}\Big(n - m - \big(S(n) - S(m)\big)\Big) =$

$$= \frac{1}{9}\left(\sum_{i=1}^{l}(\alpha_i - \beta_i) \cdot 10^i - \sum_{j=1}^{l}(\alpha_j - \beta_j) \cdot 10^j\right) = \frac{1}{9}\left(\sum_{i=1}^{l}(\alpha_i - \beta_i) \cdot \big(10^i - 1\big)\right) \ge$$

$$\geq \frac{1}{9}\left((10^l - 1) - 9 \cdot \sum_{i=0}^{l-1}(10^i - 1)\right) = \frac{1}{9}\left(10^l - 1 - 9 \cdot (1 + 10 + \cdots + 10^{l-1} - l)\right) =$$

$$= \frac{1}{9}\left(10^l - 1 - 9 \cdot \left(\frac{10^l - 1}{10 - 1} - l\right)\right) = \frac{1}{9}(10^l - 1 - 10^l + 1 + 9l) = l.$$

So,

$$A(n) - A(m) \geq l.$$

Theorem 1 is proved.

From the condition that $m < n$ of theorem 1 it follows that $A(m) + l \leq A(n),$, where $l \geq 0$. Therefore $A(m) \leq A(n)$. This shows that the following are true

Corollary 1. The function $A(n)$ is non-decreasing.

3. The rank of the numbers. Let's denote through $A^{(s)}(n) = \underbrace{A(A \ldots (A}_{s}(n)) \ldots).$

Definition. The smallest natural number s such that is called $A^{(s)}(n) = 0$ the rank of the number n: $r(n) = s$, $n \in N_0$. Let's note that

$$A(n) < \frac{n}{9}, \qquad A^{(2)}(n) < \frac{n}{9^2}, \ldots, \qquad A^{(s)}(n) < \frac{n}{9^s}.$$

Therefore, for a natural number s such that $9^s > n$, we get $A^{(s)}(n) = 0.$. This shows that any number n, $n \in N_0$ has a positive finite rank.

Theorem 2. The rank function $r(n)$ – is non-decreasing.

Proof. Let $n_1, n_2 \in$ N0 and $n_1 < n_2$.

Next, let $r(n_2)$ = s, i.e. $A^{(s)}(n_2) = 0$. Using corollary 1, the function $A(n)$ – is a non-decreasing function and using this property we repeatedly obtain

$$A(n_1) \leq A(n_2)$$

$$A(A(n_1)) \leq A(A(n_2))$$

$$\ldots \ldots \ldots$$

$$A^{(s)}(n_1) \leq A^{(s)}(n_2) = 0.$$

Therefore $A^{(s)}(n_1) = 0.$

Therefore $r(n_1) \leq r(n_2)$. So, from the condition that $n_1 \leq n_2$ it follows that

$$r(n_1) \leq r(n_2).$$

This shows that the rank function of the number $r(n)$ – is non-decreasing.

Theorem 3. Let $r(10^m) = s$, where $m \in N$. Then $r(10^{n+1} - 1) \geq s + 1.$

Proof. From the condition of theorem 3 it follows that

$$A^{(s)}(10^m) = 0 \quad \text{and} \quad A^{(s-1)}(10^m) > 0.$$

If $m = 1$, then the statement of the theorem is obvious.

Further, let $m \geq 2$.

$$A(10^{m+1}-1) = A\left(\underbrace{99\ldots9}_{m+1}\right) = \frac{1}{9}\left(\underbrace{99\ldots9}_{m+1} - 9(m+1)\right) = \underbrace{11\ldots1}_{m+1} - (m+1) > 10^m.$$

Then, by virtue of the non-decreasing function $A(n)$ (corollary 1), we obtain that

$$A^{(2)}(10^{m+1}-1) \geq A(10^m),$$

$$A^{(3)}(10^{m+1}-1) \geq A^{(2)}(10^m),$$

$$\ldots\ldots\ldots$$

$$A^{(s)}(10^{m+1}-1) \geq A^{(s-1)}(10^m) > 0$$

So

$$A^{(s)}(10^{m+1}-1) > 0,$$

Therefore

$$r(10^{m+1}-1) \geq s+1.$$

Theorem 3 is proved.

Corollary 3. If $n \to +\infty$, then rank $r(n) \to +\infty$.

Definition. The smallest number of a given rank s is denoted by. m_s. Accordingly, the largest number of a given rank s is denoted by M_s.

It follows from the definition that $M_s + 1 = m_{s+1}$, where $s = 1,2,\ldots$

Using a Java program, I made a table of numbers m_s and M_s, where $1 \leq s \leq 15$:

Table 1.

s	m_s	M_s
1	0	9
2	10	99
3	100	909
4	910	8199
5	8200	73 819
6	73 820	664 399
7	664 400	5 979 639
8	5 979 640	53 816 799
9	53 816 800	484 351 229
10	484 351 230	4 359 161 099
11	4 359 161 100	39 232 449 949
12	39 232 449 950	353 092 049 599
13	353 092 049 600	3 177 828 446 459
14	3 177 828 446 460	28 600 456 018 189
15	28 600 456 018 190	

For example, numbers 10, 909, 73819, from this table, are a-self numbers.

This fact makes it difficult to find a general formula for m_s and M_s. So, I propose the following problem:

Problem. Find a formula that expresses the values for m_s and M_s, where

$$s = 1, 2, ...$$

Let $n = 10^s$ and $r(n)$ be its rank. With the help of a program in Java, I got the following results.

s	$r(n)$	s	$r(n)$
$1 \le s \le 22$	s + 1	$128 \le s \le 148$	s + 7
$23 \le s \le 43$	s + 2	$149 \le s \le 168$	s + 8
$44 \le s \le 64$	s + 3	$169 \le s \le 189$	s + 9
$65 \le s \le 85$	s + 4	$190 \le s \le 210$	s + 10
$86 \le s \le 106$	s + 5	$211 \le s \le 231$	s + 11
$107 \le s \le 127$	s + 6		

On the table, the pattern of the rank of numbers 10^s, $s \in N$, is clearly visible. Therefore, we formulate the following

Hypothesis. Let $n = 10^s$, where $s \in N$. Let $2 + 21t \le s \le 22 + 21t$, where $t \in N$.

Then the rank of the number n:

$$r(n) = s + t + 1.$$

In conclusion, I express my gratitude to the scientific supervisor Vice-President of the National Academy of Sciences under the President of the Republic of Kazakhstan, Academician of the National Academy of Sciences of the Republic of Kazakhstan, A.S. Dzhumadildayev for setting the task and for constant attention during the writing of this work.

REFERENCES

1 Kaprekar, D.R.(1949). “Another Solitaire Game”. Scripta Mathematica. 15: 244-245.

2 Weisstein, Eric W. “Kaprekar Number”. Math World.

3 Gunjikar, K.R; Kaprekar, D.R. (1939). “Theory of Demlo numbers”(PDF). J.Univ.Bombay. VIII(3): 3-9.

4 Garner, M., “Time travel and other mathematical bewilderments”, New Yrok, 1988, 115-117.

5 Recaman, B., and Bange, D.W., Columbian numbers, American Mathematical Monthly 81(1974), 407-409.

А. ЖАРБОЛОВ

Республикалық физика-математика мектебі, Алматы қ., Қазақстан
E-mail: zharbolov@gmail.com

А-ФУНКЦИЯ САНДАРДЫҢ РАНГІСІ

Белгілі индиялық математик Д.Капрекар жаңа сандарды алудың әдісін тапты: $K(n) = n + S(n)$ *мұнда* $S(n)$ *–* n *санының ондық жүйедегі цифрларының қосындысы. Д.Капрекарға сүйене отырып бұл еңбекте бүтін сандарды алудың жаңа әдісі көрсетілген:*

$$A(n) = \frac{1}{9}\big(n - S(n)\big).$$

$A(n)$ функциясының кемімейтіндігі дәлелденген. $K(n)$ функциясына сәйкес бұл жұмыста а-туындаған және а-өзіндік туындаған сандар ұғымдары анықталған. Сандардың рангісі зерттеліп, ранг функциясының кемімейтіндігі дәлелденді. $n = 10^s$ сандарының рангісінің формуласы табылған.

***Түйін сөздер**: Д.Р.Капрекар, өзіндік туындаған сандар, репьюниттер, А-функция.*

А. ЖАРБОЛОВ

Республиканская физико-математическая школа, г. Алматы, Казахстан.
E-mail: zharbolov@gmail.com

РАНГ ЧИСЕЛ А-ФУНКЦИЙ

Известный индийский математик Д. Капрекар придумал способ получения новых чисел $K(n) = n + S(n)$, где $S(n)$ – сумма цифр числа n в десятичной записи. Следуя Д.Капрекару, в работе представлен новый способ получения чисел:

$$A(n) = \frac{1}{9}\big(n - S(n)\big).$$

Доказано, что функция $A(n)$ неубывающая. Аналогично функции $K(n)$ определены понятия а-порожденных и а-самопорожденных чисел. Исследован ранг чисел и доказано, что функция ранга – неубывающая. Найдена формула ранга для чисел $n = 10^s$.

***Ключевые слова**: Д.К.Капрекар, самопорожденные числа, А – функция.*