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EXACT SOLUTIONS OF THE TWO-DIMENSIONAL NONLINEAR SCHRODINGER EQUATION

The nonlinear Schrödinger equation as a classical model in physics is gradually becoming an urgent problem that needs to be studied. Obtaining an exact solution of the corresponding model can not only provide theoretical support for the experiment but also provide a basis for solving practical problems.

In this work, we study the propagation of waves of the two-dimensional nonlinear Schrödinger equation in nonlinear media with dispersion processes. The tan-cot function method is applied. This method is effective in solving nonlinear equations of mathematical physics. Various solutions in the form of periodic waves are obtained. Graphs of the solutions arepresented.

Keywords*:exact solutions, tan-cot function method, two-dimensional, nonlinear Schrödingerequation, ordinary differential equation, partial differential equation*

1.Introduction. Nonlinear partial differential equations (NPDEs) can transform nonlinear problems in the real world into mathematical models [1-2]. Thus the method for deriving exact solutions for the governing equating has developed, namely the Darboux transformation [3], the Hirota method [4-5], the tanh method [5], the sine-cosine method [6-7], and others.

In this work, we investigate the two-dimensional nonlinear Schrödinger equation [8], which is

$$iq_t + a_1 q_{xx} + a_2 q_{xy} + a_3 q_{yy} + i(b_1|q|^2 q)_x + ib_2(|q|^2 q_x) = 0, \tag{1}$$

where $q(x, y, t)$ is complex function of the spatial coordinates x, y and the time t, a_k, (k = 1, 2, 3) and b_j, (j = 1, 2) are real constants.

The two-dimensional NLS equation (1) is presented in [8] where optical soliton solutions are obtained by the ansatz method. However, it is noted that the tan-cotfunction method [9-10] for the NLS equation (1) was not applied. So, in this research, we find new kinds of solutions for Eq. (1) by applying the tan-cotfunction method.

2. Description of the tan-cotfunction method. The below describes the tan-cot function method [9-10] for the NPDEs in the three independent variables:

$$E\left(u_t, u_x, u_y, u_{xx}, u_{yy}, u_{xy}, u_{xxx}, \ldots\right) = 0, \tag{2}$$

where the function $u(x, y, t)$ is unknown. E is polynomial in $u(x, y, t)$ as well as its several partial derivatives including nonlinear terms and the highest order derivative. To determine the exact solutions of Eq. (1), introduce the traveling wave transformation given as

$$u(x, y, t) = u(\xi), \qquad \xi = (x + y - ct). \tag{3}$$

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Inserting the above transformation into Eq. (2), yields the ordinary differentialequation (ODE) given in the following way:

$$E(u, u', u", u''', \dots) = 0. \tag{4}$$

If required, one integrates Eq. (4) asnumerous times as possible, for simplicity, keeping the constant of integration at zero. The solutions of ODE (4) take the following form

$$u(x, y, t) = \lambda\, tan^{\beta}(\mu\xi), \tag{5}$$

or

$$u(x, y, t) = \lambda\, cot^{\beta}(\mu\xi)\,, \tag{6}$$

where $\xi = x + y - ct,$ the parameters μ, λ and β will be defined. The derivatives of (5)have forms

$$u' = \lambda\beta\mu\, tan^{\beta-1}(\mu\xi) + \lambda\beta\mu\, tan^{\beta+1}(\mu\xi), \tag{7}$$

$$u'' = \lambda\mu^2\beta(\beta - 1)\, tan^{\beta-2}(\mu\xi) + 2\lambda\mu^2\beta^2\, tan^{\beta}(\mu\xi) + \lambda\mu^2\beta(\beta + 1)\, tan^{\beta+2}(\mu\xi), \tag{8}$$

andthederivativesof(6) become

$$u' = -\lambda\beta\mu\, cot^{\beta-1}(\mu\xi) - \lambda\beta\mu\, cot^{\beta+1}(\mu\xi), \tag{9}$$

$$u'' = \lambda\mu^2\beta(\beta - 1)\, cot^{\beta-2}(\mu\xi) + 2\lambda\mu^2\beta^2\, cot^{\beta}(\mu\xi) + \lambda\mu^2\beta(\beta + 1)\, cot^{\beta+2}(\mu\xi), \tag{10}$$

and others. Using (5)-(10) into the ODE(4) we get a trigonometric equation of $tan^r(\mu\xi)$ or $cot^r(\mu\xi)$ terms. Then, we define the parameters by first balancing the exponents of each pair of functions tangent or cotangent to determine β. After, wecollect all coefficients of the same power in $tan^r(\mu\xi)$ or $cot^r(\mu\xi)$, where these coefficients mustvanish. The system of algebraic equations among the unknown λ and μ will be given and from that, we can definecoefficients.

3. Application of the tan-cot method. We assume the solutions to Eq. (1) in the form

$$q(x, y, t) = e^{i(ax+by+dt)}u(\xi),\ and \quad \xi = x + y - ct\,, \tag{11}$$

where the quantities a, b, d, c are non-zero constants. Inserting (11) intoEq. (1), separating the real and imaginary parts, we get:

$$(-d - a_1a^2 - a_2ab - a_3b^2)u + (a_1 + a_2 + a_3)u'' - (b_1a + ab_2)u^3 = 0, \tag{12}$$

$$(-c + 2a_1a + a_2b + a_2a + 2a_3b)u' + (b_1 + \tfrac{b_2}{3})(u^3)' = 0. \tag{13}$$

From Eq.(13) we can found

$$c = (2a_1 + a_2)a + (2a_3 + a_2)b, \qquad b_1 = -\tfrac{b_2}{3}. \tag{14}$$

And then in next subsection,we study Eq. (12) by the tan-cot function method

$$(-d-a_1a^2-a_2ab-a_3b^2)u+(a_1+a_2+a_3)u''-(b_1a+ab_2)u^3=0. \quad (15)$$

3.1 The tangent solution. According to method the solution of the (15) can be found by transformation

$$u_1(x,y,t)=\lambda\, tan^{\beta}(\mu\xi). \quad (16)$$

To find the tangentfunction solution we use (16) and it's derivative (8). Inserting(16) and (8) into (15)we obtain

$$\begin{gathered}(-d-a_1a^2-a_2ab-a_3b^2)\lambda\, tan^{\beta}(\mu\xi)+(a_1+a_2+a_3)(\lambda\mu^2\beta(\beta-1)\, tan^{\beta-2}(\mu\xi)+\\ +2\lambda\mu^2\beta^2\, tan^{\beta}(\mu\xi)+\lambda\mu^2\beta(\beta+1)\, tan^{\beta+2}(\mu\xi))-(b_1a+ab_2)\lambda^3\, tan^{3\beta}(\mu\xi)=0.\end{gathered} \quad (17)$$

Applying the balance method, by equating the exponents of tan^j, from (17) we define β:

$$3\beta=\beta+2,\Rightarrow\ \beta=1. \quad (18)$$

Substitute (18) in (17) we getthe next equation

$$\begin{gathered}(-d-a_1a^2-a_2ab-a_3b^2)\lambda\, tan(\mu\xi)+(a_1+a_2+a_3)(2\lambda\mu^2\, tan(\mu\xi)+\\ +2\lambda\mu^2\, tan^3(\mu\xi))-(b_1a+ab_2)\lambda^3\, tan^3(\mu\xi)=0.\end{gathered} \quad (19)$$

Equating the coefficients of each pair of the tangent function, we find the next system of algebraic equations:

$$\begin{aligned} tan(\mu\xi):&\quad \lambda(-d-a_1a^2-a_2ab-a_3b^2)+2\lambda\mu^2(a_1+a_2+a_3)=0,\\ tan^3(\mu\xi):&\quad 2\lambda\mu^2(a_1+a_2+a_3)-(b_1a+ab_2)\lambda^3=0.\end{aligned} \quad (20)$$

The system (20) give us

$$\lambda=\pm\sqrt{\frac{d+a_1a^2+a_2ab+a_3b^2}{(b_1+b_2)a}},\ \mu=\pm\sqrt{\frac{d+a_1a^2+a_2ab+a_3b^2}{2(a_1+a_2+a_3)}}. \quad (21)$$

Substituting the parameters (21) into Eq. (16) and then in (11) we have the tanget function solution of the two-dimensional NLS equation (1) in the following form

$$q_1(x,y,t)=\pm\, e^{i(ax+by+dt)}\sqrt{\frac{d+a_1a^2+a_2ab+a_3b^2}{(b_1+b_2)a}}\, tan\left(\sqrt{\frac{d+a_1a^2+a_2ab+a_3b^2}{2(a_1+a_2+a_3)}}(x+y-ct)\right), \quad (22)$$

where $c=(2a_1+a_2)a+(2a_3+a_2)b,\quad b_1=-\frac{b_2}{3}$. The dynamics of the solution $q_1(x,y,t)$ is presented in Fig. 1.

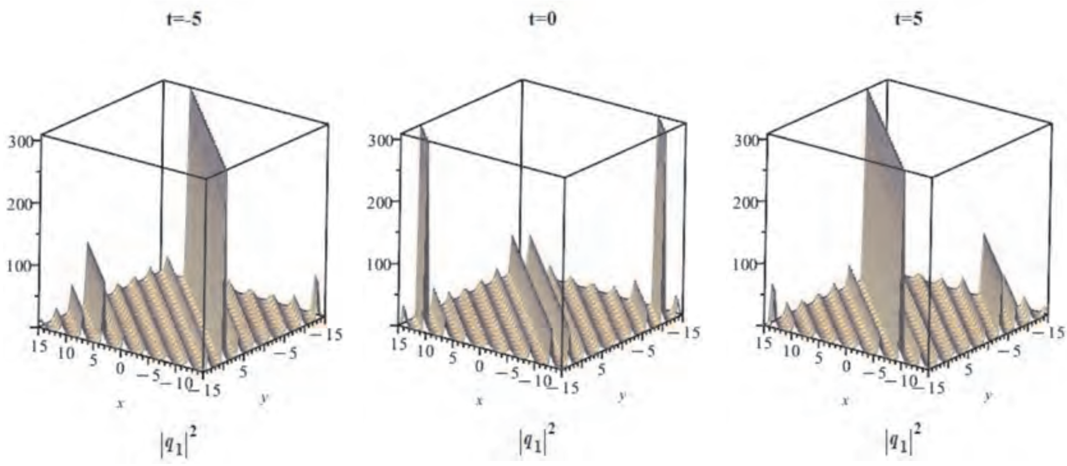

Figure 1 **–** Dynamics of thesolutions of $q_1(x, y, t)$
with the parameters: $a = 1, b = 1, d = 1, c = 6, b_1 = 1, b_2 = -\frac{1}{3}$..

3.2 The cotangent solution. According to method the solution of the (15) can be found by transformation

$$u_2(x, y, t) = \lambda \, cot^{\beta}(\mu\xi). \tag{23}$$

To find the cotangent function solution we use (23) and it's derivative (10). Inserting (23) and (10) into (15) we obtain

$$\begin{gathered}(-d - a_1 a^2 - a_2 ab - a_3 b^2)\lambda \, cot^{\beta}(\mu\xi) + (a_1 + a_2 + a_3)(\lambda\mu^2\beta(\beta - 1) \, cot^{\beta-2}(\mu\xi) + \\ +2\lambda\mu^2\beta^2 \, cot^{\beta}(\mu\xi) + \lambda\mu^2\beta(\beta + 1) \, cot^{\beta+2}(\mu\xi)) - (b_1 a + ab_2)\lambda^3 \, cot^{3\beta}(\mu\xi) = 0.\end{gathered} \tag{24}$$

Applying the balance method, by equating the exponents of cot^j, from (24) we define β:

$$3\beta = \beta + 2, \Rightarrow \ \beta = 1. \tag{25}$$

Substitute (25) in (24) we get the next equation

$$\begin{gathered}(-d - a_1 a^2 - a_2 ab - a_3 b^2)\lambda \, cot(\mu\xi) + (a_1 + a_2 + a_3)(2\lambda\mu^2 \, cot(\mu\xi) + \\ +2\lambda\mu^2 \, cot^3(\mu\xi)) - (b_1 a + ab_2)\lambda^3 \, cot^3(\mu\xi) = 0.\end{gathered} \tag{26}$$

Equating the coefficients of each pair of the cotangent function, we find the next system of algebraic equations:

$$\begin{aligned} cot(\mu\xi): &\quad \lambda(-d - a_1 a^2 - a_2 ab - a_3 b^2) + 2\lambda\mu^2(a_1 + a_2 + a_3) = 0, \\ cot^3(\mu\xi): &\quad 2\lambda\mu^2(a_1 + a_2 + a_3) - (b_1 a + ab_2)\lambda^3 = 0. \end{aligned} \tag{27}$$

The system (27) give us

$$\lambda = \pm\sqrt{\frac{d+a_1a^2+a_2ab+a_3b^2}{(b_1+b_2)a}},\ \mu = \pm\sqrt{\frac{d+a_1a^2+a_2ab+a_3b^2}{2(a_1+a_2+a_3)}}. \qquad (28)$$

Substituting the parameters (28) into Eq. (23) and then in (11) we have the cotanget function solution of the two-dimensional NLS equation (1) in the following form

$$q_2(x,y,t) = \pm\, e^{i(ax+by+dt)}\sqrt{\frac{d+a_1a^2+a_2ab+a_3b^2}{(b_1+b_2)a}}\cot\left(\sqrt{\frac{d+a_1a^2+a_2ab+a_3b^2}{2(a_1+a_2+a_3)}}(x+y-ct)\right), \quad (29)$$

where $c = (2a_1 + a_2)a + (2a_3 + a_2)b$, $b_1 = -\frac{b_2}{3}$. The dynamics of the solution $q_2(x, y, t)$ is presented in Fig. 2.

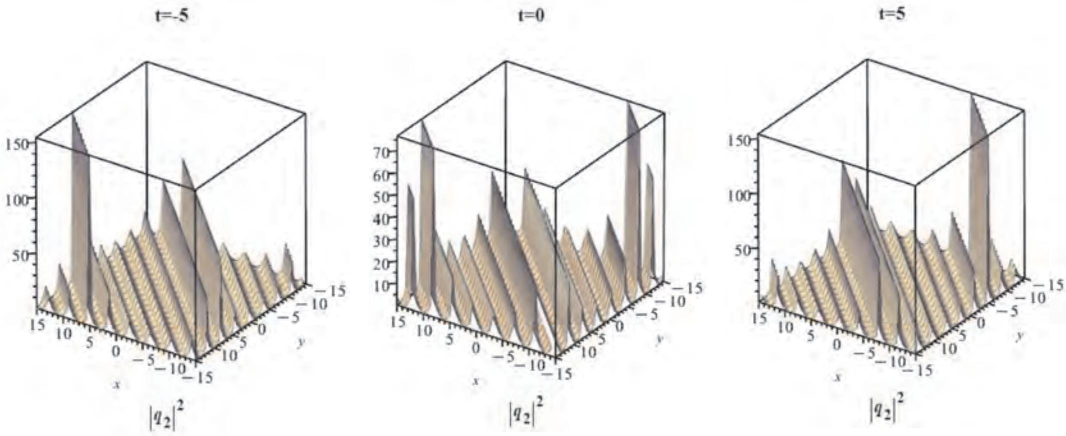

Figure 2 – Dynamics of thesolutions of $q_2(x, y, t)$
with the parameters: $a = 1, b = 1, d = 1, c = 6, b_1 = 1, b_2 = -\frac{1}{3}$.

4. Conclusion. The tan-cot function methods are applied to obtain the exact solutions of the two-dimensional NLS equation. The obtained solutions can have an application to some practical physical problems. The used method is applicable to a large variety of nonlinear partial differential equations. Furthermore, some graphs of the propagation waves are presented by choosing the specific values of the parameters.

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ЕКІ ӨЛШЕМДІ СЫЗЫҚТЫ ЕМЕС ШРЕДИНГЕР ТЕҢДЕУІНІҢ НАҚТЫ ШЕШІМДЕРІ

Сызықтық емес Шредингер теңдеуі физикадағы классикалық модель ретінде зерттеуді қажет ететін өзекті мәселе болып табылады. Тиісті модельдің нақты шешімін алу эксперментке теориялық қолдау көрсетіп қана қоймай, практикалық мәселелерді шешуге негіз бола алады.

Бұл жұмыста дисперсиясы бар сызықты емес ортада екі өлшемді сызықты емес Шредингер теңдеуінің толқындарының таралуын зерттейді. Нақты шешімдерді табу үшін тангенс және котангенс функциялары әдісі қолданылады. Бұл әдіс математикалық физиканың сызықтық емес теңдеулерін шешуде тиімді. Периодты толқындар түрінде әртүрлі шешімдер алынды. Шешім графиктері ұсынылған.

***Түйін сөздер**: нақты шешімдер,тангнес-котангнес әдісі, екі өлшемді, сызықтық емес Шредингер теңдеуі, қарапайым дифференциалдық теңдеу, дербес туынды дифференциалдық теңдеу.*

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ТОЧНЫЕ РЕШЕНИЯ ДВУМЕРНОГО НЕЛИНЕЙНОГО УРАВНЕНИЯ ШРЕДИНГЕРА

Нелинейное уравнение Шредингера как классическая модель в физике является актуальной проблемой, требующей изучения. Получение точного решения соответствующей модели может не только обеспечить теоретическую поддержку эксперимента, но и дать основу для решения практических задач.

В данной работе исследуется распространение волн двумерного нелинейного уравнения Шредингера в нелинейной среде с дисперсией. Для поиска точных решений применен метод функций тангенса и котангенса. Этот метод является эффективным при решении нелинейных уравнений математической физики. Получены различные решения в виде периодических волн. Представлены графики решений.

Ключевые слова: *точные решения, метод тангенса-котангенса, двумерное, нелинейное уравнение Шрёдингера, обыкновенное дифференциальное уравнение, дифференциальное уравнение в частных производных.*