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ALONG THE PATH OF THE GREAT KAPREKAR: A-FUNCTION, REPUNITS AND THEIR PROPERTIES

Following the famous Indian mathematician D. Kaprekar, in the paper[5], the author presented a new method for obtaining integers $A(n) = \frac{1}{9}\left(n - S(n)\right)$ where $S(n)$ - is the sum of digits of the number n In decimal notation. This A-function turned out to be related to a remarkable class of numbers - the class of integers R_n repunit. In the paper, new properties are found R_n. The properties A-function. It is proved that the set of a-self numbers is infinite and each a-self number has exactly 10 generators. The hypothesis about the distribution of a-self numbers is justified and formulated. The hypothesis about the distribution of the number of chains of «neighboring» a-self numbers is justified and formulated and the complete consistency of the two hypotheses is proved. Formulas for the chains of «neighboring» a-self numbers are found. The multiple relations between the Kaprekar function $K(n)$ and the function introduced by us $A(n)$. We studied and found all solutions of the functional equation $A(qn) = qA(n)$.

Key words: *D.R. Kaprekar, self numbers, repunits, A-function.*

1. Introduction. Indian mathematician D.R.Kaprekar discovered several remarkable classes of natural numbers such as Kaprekar's Constant[1], Kaprekar numbers[2], Harshad numbers, Demlo numbers[3].

Another outstanding discovery of D. Kaprekar is the class of self-generated numbers. It is described by the famous popularizer of science Martin Gardner in his book "Time Travel"[4]. Let's choose any natural number n and add to it the sum of its digits $S(n)$. The resulting number $K(n) = n + S(n)$ is called the generated number, and the original number is called its generator. Following D. Kaprekar, the author in the paper[5] introduced a new way of obtaining integers. The difference of a number and the sum of its numbers $n - S(n)$ is always divisible by 9. Therefore, let us define the new function for obtaining integers as follows: $A(n) = \frac{1}{9}\left(n - S(n)\right)$. Similarly to Kaprekar, for the case of the function $A(n)$ the

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classes of a-generated and a-self numbers are defined. The values $A(n)$ are directly related to the class of numbers R_n - repunit. Properties of the class of natural numbers R_n - repunit are well studied[6], [7], [8], [9]. In this paper, 2 more new properties are proved R_n (Theorems 3-1 and 3-2), and these new properties R_n will be necessary for further investigation of the properties of the function $A(n)$ and classes of numbers: a-generated and a-self numbers.

During the work, 4 hypotheses are formulated.

2. Notations. Let $N = \{1, 2, 3, ...\}$ - is the set of natural numbers. Let us denote by $N_0 = N \cup \{0\}$ - the set of non-negative integers. Throughout this paper the number system is decimal. Let $n \in N$ and

$$n = \alpha_0 + \alpha_1 \cdot 10 + \cdots + \alpha_k \cdot 10^k = \sum_{i=1}^{k} \alpha_i \cdot 10^i, \quad where \quad \alpha_k \neq 0.$$

Further, the number $d(n) = k + 1$ let us call the order of the number n. Hence, the order of a number n shows the number of digits of the number n in the decimal system. The sum of digits of a number n denote by

$$S(n) = \alpha_0 + \alpha_1 + \cdots + \alpha_k = \sum_{i=0}^{k} \alpha_i.$$

In the work, the Greek letters α_i, β_i, γ_i to denote numbers:

$0 \leq \alpha_i \leq 9;$ $\quad 0 \leq \beta_i \leq 9;$ $\quad 0 \leq \gamma_i \leq 9,$ where $i \in N_0$.

3. Repunits and their properties. Repunites are natural numbers, the record of which in the decimal number system consists of only one unit.

$R_1 = 1, R_2 = 11, R_3 = 111$ and e.t.c.:

General view of the repunit $R_n = \frac{10^n - 1}{9}, n = 1, 2, 3, ...$

Repunites are related by a recurrence relation:

$$R_n = 10 \cdot R_{n-1} + 1, \qquad n = 2, 3, 4 ...$$

or (3–1)

$$R_n - 10 \cdot R_{n-1} = 1, \qquad n = 2, 3, 4 ...$$

Repunites are a very interesting class of numbers that have been investigated by many famous mathematicians [6], [7], [8], [9] and studied for their remarkable properties.

Consider the following numbers, which we will need in our further work.

$$B_{n,l} = R_n - 9 \cdot \sum_{i=l}^{n-1} R_i,$$

$$C_{n,l} = 8 \cdot R_n + 9 \cdot \sum_{i=l}^{n-1} R_i,$$

where n=2, 3, ..., 1≤ l ≤n-1.

It's obvious that

$$B_{n,l} + C_{n,l} = 9 \cdot R_n = 9 \cdot \frac{10^n - 1}{9} = 10^n - 1. \quad (3\text{–}2)$$

Let's define some properties of these numbers.

Motion 3-1.

$$B_{n,1} = R_n - 9 \cdot \sum_{i=1}^{n-1} R_i = n \text{ where } n = 2, 3, \ldots \quad (3\text{–}3)$$

The **proof** is carried out by induction on *n*.

Using the results of Proposition (3-1) we prove the following theorem (3-1):

Theorem 3-1. $B_{n,l} = R_l + n - l,$ where $n = 2, 3, \ldots, \quad 1 \le l \le n - 1$

Proof.

$$B_{n,l} = R_n - 9 \cdot \sum_{i=l}^{n-1} R_i = R_n - 9 \cdot \sum_{i=1}^{n-1} R_i + 9 \cdot \sum_{i=1}^{l-1} R_i = n + 9 \cdot (R_1 + R_2 + \cdots + R_{l-1}) =$$

$$= n + 9 \cdot \left(\frac{10 - 1}{9} + \frac{10^2 - 1}{9} + \cdots + \frac{10^{l-1} - 1}{9} \right) =$$

$$= n + \left(10 + 10^2 + \cdots + 10^{l-1} - (l - 1) \right) = n + \frac{10 \cdot \left(10^{l-1} - 1\right)}{9} - l + 1 =$$

$$= n + 10 \cdot R_{l-1} + 1 - l = n + R_l - l = R_l + n - l.$$

Theorem 3-1 is proved.

Theorem 3-2.

$$C_{n,l} = 10^n - R_l - n - 1 + l, \text{ where } n = 2, 3, \ldots, \qquad 1 \le l \le n - 1.$$

Proof. Let's use formula (3-2) and theorem (3-1):

$$C_{n,l} = 10^n - 1 - B_{n,l} = 10^n - 1 - (R_n + n - l) = 10^n - R_l - n - 1 + l$$

Theorem 3-2 is proved.

The following follows from Theorem 3-1

Corollary 3-1. $B_{n,2} = n + 9$

Proof. $B_{n,2} = R_2 + n - 2 = 11 + n - 2 = n + 9.$

From Theorem 3-2 we easily obtain the following

Investigation 3-2. $C_{n,1} < 10^n$

Proof. $C_{n,1} = 10^n - R_1 - n - 1 + 1 = 10^n - 1 - n < 10^n.$

Proposal 3-2. For any $s, t \in N$ the following identity holds:

$$R_{s+t} = 10^t \cdot R_s + R_t$$

Proof. $R_{s+t} = \underbrace{11\ldots1}_{s+t} = \underbrace{11\ldots1}_{s}\underbrace{00\ldots0}_{t} + \underbrace{11\ldots1}_{t} = 10^t \cdot R_s + R_t.$

Proposition 3-2 is proved. From Proposition 3-2 we obtain the following

Corollary 3-3. $R_{s+t} - 10^t \cdot R_s = R_t.$ (3–4)

4. Function A(n) and its properties. The difference of a number and the sum of its digits *n* – *S*(*n*) is a multiple of 9. Therefore, a new function for obtaining integers is introduced in the paper[5].

$$A(n) = \frac{1}{9}\big(n - S(n)\big) \text{ where } n \in N,\ A(n) \in N_0$$

Then

$$A(n) = \frac{1}{9}\left(\sum_{i=0}^{k} \alpha_i \cdot 10^i - \sum_{i=0}^{k} \alpha_i\right) =$$

$$= \frac{1}{9}\sum_{i=1}^{k} \alpha_i \left(10^i - 1\right) = \sum_{i=1}^{k} \alpha_i \cdot \left(\frac{10^i - 1}{9}\right) = \sum_{i=1}^{k} \alpha_i R_i\,.$$

Then,

$$A(n) = \alpha_1 \cdot R_1 + \alpha_2 \cdot R_2 + \cdots + \alpha_k \cdot R_k = \sum_{i=1}^{k} \alpha_i R_i \qquad (4\text{–}1)$$

From this formula we formulate a property of the function *A*(*n*):

Proposition 4-1. The value *A*(*n*) does not depend on α_0 units of the number *n*.

Let's note that.

$$A\left(10^i\right) = R_i \text{ and } A\left(\alpha_i 10^i\right) = \alpha_i R_i \qquad i = 1, 2, \ldots$$

Now using formula (4-1) we note one more property *A*(*n*).

Proposition 4-2. The function *A*(*n*) is additive on the digits of the number $n \in N_0$:

$$A\left(\alpha_k \cdot 10^k + \cdots + \alpha_1 \cdot 10 + \alpha_0\right) = \alpha_k R_k + \cdots + \alpha_1 R_1 =$$

$$= A\left(\alpha_k \cdot 10^k\right) + \cdots + A(\alpha_1 \cdot 10) + A(\alpha_0), \quad where \quad A(\alpha_0) = \frac{1}{9}(\alpha_0 - \alpha_0) = 0.$$

Theorem 4-1. Let $d(n) = k + 1$. Then $R_k \le A(n) \le R_{k+1} - k - 1.$

Proof. By formula (4-1)

$$A(n) = \sum_{i=1}^{k} \alpha_k R_k\,. \text{ Therefore}$$

1) $R_k \le A(n)$

2) $$A(n) \le 9 \cdot \sum_{i=1}^{k} R_i = 9 \cdot \sum_{i=1}^{k} \frac{10^i - 1}{9} = \sum_{i=1}^{k} \left(10^i - 1\right) =$$

$$= \sum_{i=1}^{k} 10^i - k = \frac{10 \cdot \left(10^k - 1\right)}{10 - 1} - k = \frac{10^{k+1} - 1 - 9}{9} - k =$$

$$= \left(\frac{10^{k+1} - 1}{9}\right) - 1 - k = R_{k+1} - k - 1.$$

Theorem 4-1 is proved.

Proposition 4-3. Let $n = \alpha_s \cdot 10^s, \quad m = \beta_s \cdot 10^s.$ Then

1) if $\alpha_s + \beta_s \le 9,$ then $A(n+m) = A(n) + A(m).$

2) if $10 \le \alpha_s + \beta_s \le 18,$ then $A(n+m) = A(n) + A(m) + 1.$

Proof.

1) Let $\alpha_s + \beta_s \le 9.$ Then

$$A(n+m) = A(\alpha_s \cdot 10^s + \beta_s \cdot 10^s) = A\left((\alpha_s + \beta_s) \cdot 10^s\right) =$$

$$= (\alpha_S + \beta_S) \cdot R_S = \alpha_s \cdot R_s + \beta_s \cdot R_s =$$

$$= A(\alpha_s \cdot 10^s) + A(\beta_s \cdot 10^s) = A(n) + A(m).$$

2) Let $10 \le \alpha_s + \beta_s \le 18$. Then

$$A(n+m) = A\left((\alpha_s + \beta_s) \cdot 10^s\right) = A(10 \cdot 10^s + (\alpha_s + \beta_s - 10) \cdot 10^s) =$$

$$= A(10^{s+1} + (\alpha_s + \beta_s - 10) \cdot 10^s) = R_{s+1} + (\alpha_s + \beta_s - 10) \cdot R_s$$

$$= (10R_s + 1) + \alpha_s \cdot R_s + \beta_s \cdot R_s - 10 \cdot R_s =$$

$$= A(n) + A(m) + 1.$$

Proposition 4-4 is proved.

Theorem 4-2. For any $n, m \in N$

$$A(n+m) \ge A(n) + A(m).$$

Proof. Let $d(n) = k + 1, \quad m < n.$

Then

$$n = \sum_{i=0}^{k} \alpha_i \cdot 10^i, \text{ where } \alpha_k \ne 0,$$

$$m = \sum_{j=1}^{k} \beta_j \cdot 10^j. \text{ Then,}$$

$$A(n+m) = A\left(\sum_{i=0}^{k} \alpha_i \cdot 10^i + \sum_{j=1}^{k} \beta_j \cdot 10^j\right) =$$

$$= A\left(\sum_{i=0}^{k} (\alpha_i + \beta_i) \cdot 10^i\right).$$

Let's study the sums of digits at corresponding digits (powers of 10) of numbers *n* and *m*:

$$\alpha_0 + \beta_0; \quad \alpha_1 + \beta_1; \quad \dots; \alpha_k + \beta_k.$$

If all sums are less than or equal to 9, then according to proposition (4-5)-(1)

$$A(n+m) = A(n) + A(m).$$

If at least one sum is greater than or equal to 10, then according to proposition (4-5)-(2)

$$A(n+m) > A(n) + A(m).$$

Theorem 4-2 is proved.

Proposal 4-4. For any $n_1, n_2, n_3 \in N$ the inequality holds:

$$A(n_1 + n_2 + n_3) \geq A(n_1) + A(n_2) + A(n_3)$$

Proof. Let us use Theorem (4-2) twice:

$$A(n_1 + n_2 + n_3) = A\big(n_1 + (n_2 + n_3)\big) \geq A(n_1) + A(n_2 + n_3) \geq A(n_1) + A(n_2) + A(n_3).$$

Proposition (4-4) is proved.

Similarly to Proposition (4-4) the following is proved

Theorem 4-3. Let $n_1, n_2, \cdots, n_s \in N$. Then the inequality holds:

$$A(n_1 + n_2 + \ldots + n_s) \geq A(n_1) + A(n_2) + \ldots + A(n_s)$$

5. A – generated and a – self-generated numbers. Let $A(n) = m$, where $m, n \in N_0$. A number *m* is called a - generated number, and the number *n* is called its a-generator. An a -generated number can have more than one generator. If a number *p* has no a - generators, it is called a-self number. Examples of a-self numbers are: 10, 20, 32, 109, 110, 1108, 1109, 1110 and others. It follows from formula (4-1) that any *a* – generated number *m* can be represented in the form:

$$m = \sum_{i=1}^{k} \alpha_i R_i$$

Theorem 5-1. The set of a-self numbers is infinite.

Proof. Let us show that all numbers of the form

$$p = \sum_{i=2}^{k} \alpha_i R_i - 1 \quad \text{are a – self numbers}$$

Let's prove the opposite. Let a number *m* is a generator of a number *p*. According to Proposition (4-1), we can assume that $\beta_0 = 0$. Then

$$m = \sum_{j=1}^{k} \beta_j \, 10^j$$

Then, $$A(m) = p = \sum_{i=2}^{k} \alpha_i R_i - 1 < \sum_{i=2}^{k} \alpha_i R_i = A(n), \qquad (5–1)$$

Where $$n = \sum_{i=2}^{k} \alpha_i 10^i, \qquad \alpha_0 = \alpha_1 = 0$$

From the paper[5] and inequality (5-1) it follows that $m < n$.

Given that $\alpha_0 = \alpha_1 = 0$, we obtain.

$$l(n, m) = l \geq 2$$

It follows from the paper[5] that

$$A(n) - A(m) \geq l \geq 2 \qquad (5–2)$$

So, $$p = A(m) \quad \text{and} \quad p - A(m) = 0.$$

Then.

$$\left(\sum_{i=2}^{k} \alpha_i R_i - 1 \right) - A(m) = 0, \qquad \sum_{i=2}^{k} \alpha_i R_i - A(m) = 1,$$

$$A(n) - A(m) = 1. \qquad (5–3)$$

The obtained equality (5-3) contradicts the inequality (5-2). This contradiction proves theorem (5-1).

Theorem 5-2. Every a-generated number *m* has exactly 10 a-generators,

Proof. Let

$$m = \sum_{i=1}^{k} \alpha_i R_i -$$

- is an a-self number. Let the numbers n_1 and n_2, $n_1 < n_2$ be a-generators of a number *m*:

$$m = A(n_1) = A(n_2).$$

Then $A(n_2) - A(n_1) = 0$. It follows from Theorem (5 – 1) that $d(n_2 - n_1) = 0$. This means that the numbers n_1 and n_2 in the decimal notation differ from each other only by digits in the unit digits.

Thus, the following 10 numbers are a - number generators *m*:

$$n_S = \sum_{i=1}^{k} \alpha_i 10^i + s, \text{ where } 0 \le s \le 9$$

Theorem 5-3. Let $\omega_k = 9 \cdot 10^k + 9$, where $k = 1, 2, \ldots$.

Then $A(\omega_k) = 10^k - 1$ and $A(\omega_k + 1) = 10^k$.

Proof. Let's use formula (4-1):

1. $A(\omega_k) = A\left(9 \cdot 10^k + 9\right) = 9 \cdot R_k + 9 \cdot R_0 = 9 \cdot R_k = 10^k - 1.$

2. $A(\omega_k + 1) = A\left(9 \cdot 10^k + 1 \cdot 10\right) = 9 \cdot R_k + 1 \cdot R_1 = 10^k - 1 + 1 = 10^k.$

Theorem (5-3) is proved.

From Theorem (5-3) and the paper[5] the following follows

Proposition 5-1. Let a number *n* such that

$$d\big(A(n)\big) = p, \text{ where } p = 2, 3, \ldots$$

Then

$$min\{n\} = \omega_{p-1} + 1, \qquad \{n\} = \omega_p$$

Therefore

$$9 \cdot 10^{p-1} + 10 \le n \le 9 \cdot 10^p + 9.$$

6. Distribution of a-self numbers. The number of digits in the decimal notation of a number *n* let us call the order of the number *n* and denote by *d*(*n*).

$$10^{d(n)-1} \le n \le 10^{d(n)}$$

With the help of a program in Java and Python language, having investigated all numbers up to 10^8 made the following table of distribution of a-self numbers.

Table 1.

d(*n*)	Number of all numbers of order *d*(*n*)	The number of a-self numbers of order *d*(*n*)	% a-self numbers
1	2	3	4
1	9	0	0
2	9·10	9	10

1	2	3	4
3	$9{\cdot}10^2$	$9{\cdot}10$	10
4	$9{\cdot}10^3$	$9{\cdot}10^2$	10
5	$9{\cdot}10^4$	$9{\cdot}10^3$	10
6	$9{\cdot}10^5$	$9{\cdot}10^4$	10
7	$9{\cdot}10^6$	$9{\cdot}10^5$	10
8	$9{\cdot}10^7$	$9{\cdot}10^6$	10

Based on the results of the table we can formulate the following hypothesis.

Hypothesis 1. Among all numbers of a given order. $d(n) = t,\ t \geq 2,$ there are $9 \cdot 10^{t-1}$, the number of a-self numbers is equal to $9 \cdot 10^{t-2}$. Thus, the number of a-self numbers of the order *t* is exactly 10% of the number of all numbers of order *t*.

7. Distribution of the number of chains of "neighboring" a-self numbers.

Definition. A-self number p_1 is called a unary a-self if the numbers $p_1 - 1$ and $p_1 + 1$ are a-self numbers.

Definition. If all numbers of the set

$$P_s = \{p_{s,1}, \quad p_{s,1} + 1, \quad \ldots, \ p_{S,1} + s - 1\}, \quad s \ = 2, 3, \ldots$$

are a-self numbers, and the numbers $p_{s,1} - 1$ and $p_{s,1} + s$ are a-self numbers, then all numbers of the set P_S are called a chain of a-self «neighboring» numbers of length *s*.

Consider $N_t = \{n \in N, \ \ d(n) = t\}$ - the set of all numbers of order *t*. Obviously, the number of numbers of the set N_t is equal to $9 \cdot 10^{t-1}$. According to hypothesis 1, the number of all a-self numbers is equal to the set N_t equals $9 \cdot 10^{t-2}$.

With the help of a program in Java language having studied all numbers up to 10^8 made table 2 of the distribution of the number of chains of «neighboring» a-self numbers N_t where $1 \leq t \leq 8$.

Table 2.

$d(n)$	The number of all a-self numbers.	$s = 1$	$s = 2$	$s = 3$	$s = 4$	$s = 5$	$s = 6$	$s = 7$
1	0	0	0	0	0	0	0	0
2	9	9	0	0	0	0	0	0
3	$9{\cdot}10$	72	9	0	0	0	0	0
4	$9{\cdot}10^2$	729	72	9	0	0	0	0
5	$9{\cdot}10^3$	$729 \cdot 10$	729	72	9	0	0	0
6	$9{\cdot}10^4$	$729 \cdot 10^2$	$729 \cdot 10$	729	72	9	0	0
7	$9{\cdot}10^5$	$729 \cdot 10^3$	$729 \cdot 10^2$	$729 \cdot 10$	729	72	9	0
8	$9{\cdot}10^6$	$729 \cdot 10^4$	$729 \cdot 10^3$	$729 \cdot 10^2$	$729 \cdot 10$	729	72	9

Having summarized the pattern of Table 2, we can formulate

Hypothesis 2. In the set N_t, where $t \geq 5$, is the number of «neighboring» chains a-self numbers of length *s* are as follows:

Chain lengths *s*	Number of circuits lengths *s* «neighboring» a-self numbers	Numbers of all a-self numbers, chain lengths *s*
1	$9^3 \cdot 10^{t-4}$	$1 \cdot 9^3 \cdot 10^{t-4}$
2	$9^3 \cdot 10^{t-5}$	$2 \cdot 9^3 \cdot 10^{t-5}$
...	... ...	... ...
$t-4$	$9^3 \cdot 10$	$(t-4) \cdot 9^3 \cdot 10$
$t-3$	9^3	$(t-3) \cdot 9^3$
$t-2$	72	$(t-2) \cdot 72$
$t-1$	9	$(t-1) \cdot 9$

And the total number of a-self numbers in the set N_t is equal to:

$$P(t) = (t-1) \cdot 9 + (t-2) \cdot 72 + (t-3) \cdot 9^3 + (t-4) \cdot 9^3 \cdot 10 + \cdots +$$
$$+2 \cdot 9^3 \cdot 10^{t-5} + 1 \cdot 9^3 \cdot 10^{t-4}.$$

Theorem 7-1. The following equality is true

$$P(t) = 9 \cdot 10^{t-2}, \qquad t \geq 4$$

The proof is carried out by induction on *t*.

Theorem 7-1 shows that Hypothesis 1 is in complete agreement with Hypothesis 2.

8. Formulas for chains of "neighboring" a-self numbers.

Theorem 8–1. All numbers of the form $p_1 = \sum_{i=2}^{k} \alpha_i R_i - 1$, where $\alpha_2 \neq 0$ (8–1) are single a-self numbers.

Proof.

1) It follows from Theorem (5-1) that all numbers of the form (8-1) are a-self numbers.

2-a) Let us show that all numbers of the form

$$p_1 - 1 = \sum_{i=z}^{k} \alpha_i R_i - 2, \text{ where } \alpha_2 \geq 1,$$

are a-generated numbers.

Indeed,

$$p_1 - 1 = \sum_{i=2}^{k} \alpha_i R_i - 2 = \sum_{i=3}^{k} \alpha_i R_i + (\alpha_2 - 1) R_2 + R_2 - 2 =$$
$$= \sum_{i=3}^{k} \alpha_i R_i + (\alpha_2 - 1) \cdot R_2 + 1 \cdot 9 = A(n_1),$$

where $$n_1 = \sum_{i=3}^{k} \alpha_i \cdot 10^i + (\alpha_2 - 1) \cdot 10^2 + 1 \cdot 10.$$

So, the numbers p_1 are a-generated numbers.

2-b) Now we show that all numbers of the *form* $p_1 + 1$ are a – generated numbers.

Indeed,

$$p_1 + 1 = \sum_{i=2}^{k} \alpha_i R_i = A(n_2), \text{ where } n_2 = \sum_{i=2}^{k} \alpha_i \cdot 10^i,$$

n_2 are a – number generator $p_1 + 1$, so numbers of the form $p_1 + 1$ are a – agenerated.

The theorem is proved.

Theorem 8-2. All numbers of the set

$$P_2 = \{p_{2,1} \;\; ; \;\; p_{2,1} + 1\}, \text{ where } p_{2,1} = \sum_{i=3}^{R} \alpha_i R_i - 2, \quad \alpha_3 \neq 0$$

They are a chain of "neighboring" a-self numbers of length 2.

Proof.

1-a) Let us show that the number $p_{2,1}$ is an a-self number. Let's prove the opposite.

Let $p_{2,1}$ be an a-generated number and the number m_1 – be its a -generator. Then

$$A(m_1) = p_{2,1} = \sum_{i=3}^{k} \alpha_i R_i - 2 = A(n) - 2, \text{ where } n = \sum_{i=3}^{k} \alpha_i \cdot 10^i, \qquad \alpha_3 \neq 0$$

Hence, we obtain that

$$A(n) - A(m_1) = 2 \qquad (8–2)$$

From $A(n) > A(m_1)$ and paper[5] it follows that $n > m_1$. Since $\alpha_0 = \alpha_1 = \alpha_2 = 0$, то «difference» of powers of numbers n and m1

$$l(n; m_1) \geq 3$$

Now from Theorem (4–1) it follows that

$$A(n) - A(m_1) \geq l(n; m_1) \geq 3 \qquad (8–3)$$

Equality (8-2) contradicts inequality (8-3). The contradiction proves that numbers of the form $p_{2,1}$ are a-self numbers.

1-b) It is similarly proved that the numbers of the *form* $p_{2,1} + 1$ are a-self numbers.

2-a) Let us show that the numbers $p_{2,1} - 1$ are a-generated.

Indeed,

$$p_{2,1} - 1 = \sum_{i=3}^{k} \alpha_i R_i - 3 = \sum_{i=4}^{k} \alpha_i R_i + (\alpha_3 - 1) \cdot R_3 + R_3 - 3 =$$

$$= \sum_{i=4}^{k} \alpha_i R_i + (\alpha_3 - 1)R_3 + 111 - 3 =$$

$$= \sum_{i=4}^{k} \alpha_i R_i + (\alpha_3 - 1)R_3 + 9 \cdot 11 + 9 \cdot 1 = A(n_1),$$

where $$n_1 = \sum_{i=4}^{k} \alpha_i \cdot 10^i + (\alpha_3 - 1) \cdot 10^3 + 9 \cdot 10^2 + 9 \cdot 10.$$

So, the numbers $p_{2,1} - 1$ are a-generated numbers.

2-b) Now we show that the numbers of the form $p_{2,1} + 2$ are a – agenerated numbers. Indeed,

$$p_{2,1} + 2 = \sum_{i=3}^{k} \alpha_i R_i = A(n_2), \text{ where } n_2 = \sum_{i=3}^{k} \alpha_i \cdot 10^i$$

So, numbers of the form $p_{2,1} + 2$ are a-generated numbers.

Theorem 8-2 is proved.

By examining all single a-self numbers up to 10^5 with the help of a program in Java language we can formulate the following

Hypothesis 3. There are no single a-self numbers, except for the numbers (8-1)

$$p_1 = \sum_{i=2}^{k} \alpha_i R_i - 1, \text{ where } \alpha_2 \neq 0$$

By analogy of theorem (8-2) we can formulate the following one

Hypothesis 4. All chains a-self "neighboring" numbers of length s are completely described by the multiplicities (7-1)

$$P_s = \{p_{s,1}\,;\ p_{s,1} + 1; \ldots; p_{s,1} + s - 1\}, \quad s = 2, 3, \cdots$$

where $$p_{s,1} = \sum_{i=s+1}^{k} \alpha_i R_i - s, \quad \alpha_{s+1} \neq 0$$

The truth of these hypotheses 3, 4 is tested by a Java program for a-self numbers up to 10^8 where $s = 2, 3, 4, 5, 6, 7$.

9. Multiple relationship between K(n) and A(n)

In this part of the paper, we study the multiple relations between the Kaprekar function $K(n)$ and the new function introduced by us $A(n)$. Therefore, we study and solve Eq:

$$K(n) = q \cdot A(n), \text{ where } n, q \in N \tag{9–1}$$

Let's convert this equation to its equivalent equation:

$$n + S(n) = q \cdot \frac{1}{9}\big(n - S(n)\big), \qquad 9 \cdot n + 9 \cdot S(n) = q \cdot n - q \cdot S(n),$$

$$(q+9)\cdot S(n)=(q-9)\cdot n. \quad (9\text{–}2)$$

So, equations (9-1) and (9-2) are equivalent. Therefore, to study equation (9-1), we will often study equation (9-2).

Proposition 9-1. Let $t \in N$ and $t \geq 3$.

Then the inequality is true:

$$18t < 10^{t-1}.$$

The proof is easily carried out by mathematical induction by *t*.

It is easy to establish that the following is true

Proposition 9-2. Let $q \geq 30.$ Then the following inequalities are true

$$q+9 < 2(q-9)$$

Theorem 9-1. When $q \leq 9$ and $q \geq 30$ equation (9-1) has no solutions.

1) Let $q \leq 9$. Then the left side of equation (9-2) is a positive number, and the right side is a non-negative number. Therefore, equation (9-2) has no solutions at $q \leq 9$.

2) Let's say $q \geq 30$:

a) $d(n)=1,$ i.e. $S(n)=n=\alpha_0 \neq 0.$ Then it follows from (9-2) that

$$(9+q)\cdot\alpha_0=(q-9)\alpha_0,$$
$$9+q=q-9$$
$$9=-9, \text{ contradiction.}$$

In the case of $d(n)=1$ and $q \geq 30$ equation (9-2) has no solution.

b) $d(n)=2,$ i.e. $n=\alpha_1\cdot 10+\alpha_0,\ \ \alpha_1 \neq 0.$

Then it follows from equation (9-2) that

$$(9+q)\cdot(\alpha_1+\alpha_0)=(q-9)(\alpha_1\cdot 10+\alpha_0),$$
$$9\alpha_1+9\alpha_0+q\alpha_1+q\alpha_0=q\alpha_1\cdot 10+q\alpha_0-90\alpha_1-9\alpha_0,$$
$$9\alpha_1 q-99\alpha_1-18\alpha_0=0 \quad \rightarrow \quad \alpha_1 q-11\alpha_1-2\alpha_0=0$$
$$\alpha_1(q-11)=2\alpha_0. \quad (9\text{–}3)$$

By convention $q \geq 30,$ therefore $q-11 \geq 19.$ Thus, the left side of equation (9-3) is greater than or equal to 19 and the right side is less than 18. Hence, equation (9-3) has no solution.

c) Let $d(n)=t \geq 3.$ This implies that $S(n) \leq 9\cdot t$ and $n \geq 10^{t-1}.$

Now using theorems (9-1) and (9-2) we obtain

$$(q+9)\cdot S(n) \leq (q+9)\cdot 9t=\frac{1}{2}(q+9)\cdot 18t<\frac{1}{2}(q+9)\cdot 10^{t-1}<$$
$$<\frac{1}{2}\cdot 2(q-9)\cdot 10^{t-1}=(q-9)\cdot 10^{t-1} \leq (q-9)\cdot n.$$

So, we obtain the following inequality

$$(q+9)\cdot S(n) < (q-9)\cdot n.$$

This inequality contradicts equation (9-2). Hence, in the case of $d(n) = t \geq 3$ equation (9-2) and its equivalent equation (9-1) have no solutions. Theorem (9-1) is proved.

Proposition 9-3. Let $t \geq 4$. Then the following inequality is true:

$$342(t+1) < 10^t.$$

We prove the proof by induction on *t*.

Theorem 9-2. Let $n > 10^4$ and $10 \leq q \leq 29$. Then equation (9-1) has no solutions.

Proof. Let $n > 10^t$, where $t \geq 4$. Then $S(n) \leq 9\cdot(t+1)$. Then, by proposition (9-3)

$$(q+9)\cdot S(n) \leq (29+9)\cdot 9(t+1) =$$

$$= 342\cdot(t+1) < 10^t < n < (q-9)n.$$

So, we get the following inequality:

$$(q+9)\cdot S(n) < (q-9)\cdot n \qquad (9–4)$$

The obtained inequality (9-4) contradicts equation (9-2). Therefore, equation (9-2) has no solutions at $n > 10^4$. Consequently, the equivalent equation (9-1) also has no solutions at $n > 10^4$

Theorem (9-2) is proved.

It follows from Theorem (9-1) and (9-2) that the following is true

Theorem 9-3. Equation (9-1) can have solutions only under the conditions when $n \leq 10^4$ and $10 \leq q \leq 29$..

Using a Java program, I found all the solutions to equation (9-1):

q	*n*
10	114, 133, 152, 171, 190, 209, 228, 247, 266, 285, 399
11	10, 20, 30, 40, 50, 60, 70, 80, 90
12	21, 42, 63, 84
13	11, 22,, 33, 44, 55, 66, 77, 88, 99
14	23, 46, 69
15	12, 24, 36, 48
16	25
17	13, 26, 39
18	27
19	14, 28
20	29

21	15
23	16
25	17
27	18
29	19

So, theorems (9-1), (9-2) and (9-3) give the complete solution of the multiple relation between $K(n)$ and $A(n)$:

$$K(n) = qA(n) \qquad (9\text{-}1)$$

10. Investigation and solution of the functional equation $A(qn) = qA(n)$

In this part of the paper we study and solve the functional equation

$$A(qn) = qA(n), \text{ where } n, q \in N, \quad q \geq 2 \qquad (10\text{–}1)$$

According to paper[5] the function $A(n)$ is additive over the digits of the number n:

$$A\left(\alpha_k \cdot 10^k + \cdots + \alpha_1 \cdot 10 + \alpha_0\right) =$$

$$= A\left(\alpha_k \cdot 10^k\right) + \cdots + A(\alpha_i \cdot 10) + A(\alpha_0), \text{ where } A(\alpha_0) = 0. \qquad (10\text{–}2)$$

Similarly to formula (10-2), the following is evident

Proposition 10-1. Let $n = \sum_{i=0}^{k} \alpha_i \cdot 10^i$, где $0 \leq \alpha_i \leq 9, \quad i = \overline{1,k}$. Let $\alpha_i = \beta_i + \gamma_i$, where $\beta_i \geq 0, \quad \gamma_i \geq 0, \quad i = \underline{1,k}$. Then.

$$A(n) = A\left(\sum_{i=0}^{k} \beta_i \cdot 10^i\right) + A\left(\sum_{i}^{k} \gamma_i \cdot 10^i\right). \qquad (10\text{–}3)$$

Formulas (10-2) and (10-3) will be very useful in investigating and solving the functional equation (10-1).

Definition. Let

$$d(n) = k + 1, \quad \text{i.e.} \quad n = \sum_{i=0}^{k} \alpha_i \cdot 10^i, \text{ where } \alpha_i \text{ – digits of the number } n,\ i = \overline{0,k}.$$

Let l be the number of digits of the number n, which are greater than or equal to t, $1 \leq t \leq 9$. Then the number l let us call a vertex of order t number n and denote

$$v_t(n) = l.$$

For example, if $v_5(n) = 0$, then it means that all cirphs of the number n are less than or equal to 4:

$$0 \leq \alpha_i \leq 4 \qquad i = \underline{0,k}.$$

We first study the functional equation (10-1) in the case when $q = 2$:

$$A(2n) = 2A(n), \quad n \in N. \tag{10–4}$$

Proposition 10-2. Let $d(n) = k + 1.$ Let $v_5(n) = 0.$ Then the number *n* is a solution of the functional equation (10-4).

Proof. As it was shown above, if $v_5(n) = 0,$ then

$$0 \le \alpha_i \le 4, \qquad i = \underline{0, k}.$$

Then.
$$0 \le 2\alpha_i \le 8, \qquad i = \underline{0, k}.$$

Considering (10-5) we obtain

$$A(2n) = A\left(2 \cdot \sum_{i=o}^{k} \alpha_i \cdot 10^i\right) = A\left(\sum_{L=0}^{k} (2\alpha_i) \cdot 10^i\right) =$$

$$= \sum_{i=1}^{k} (2\alpha_i) \cdot R_i = 2 \cdot \sum_{i=1}^{k} \alpha_i R_i = 2 \cdot A(n).$$

Proposition 10-2 is proved.

Proposition 10-3. Let $n = \alpha_k \cdot 10^k,$ where $5 \le \alpha_k \le 9.$ Ie. $v_5(n) = 1.$ Then

$$A(2n) = 2 \cdot A(n) + 1. \tag{10–5}$$

Proof. Let us note that $10 \le 2\alpha_k \le 18,$ and $0 \le 2\alpha_k - 10 \le 8.$ Further,

$$A(2n) = A(2\alpha_k \cdot 10^k) = A\left((10 + 2\alpha_k - 10) \cdot 10^k\right) =$$

$$= A(10^{k+1} + (2\alpha_k - 10) \cdot 10^k).$$

By virtue of property (10-2)

$$A(2n) = A(10^{k+1}) + A\left((2\alpha_k - 10) \cdot 10^k\right) =$$

$$= R_{k+1} + (2\alpha_k - 10) \cdot R_k = (10 + 1\,) + 2\alpha_k \cdot R_k - 10 \cdot R_k =$$

$$= 2(\alpha_k \cdot R_k) + 1 = 2 \cdot A(n) + 1.$$

Proposition 10-3 is proved.

Proposition 10-4. Let $n \in N$ such that $v_5(n) = l \ge 1.$ Then

$$A(2n) = 2 \cdot A(n) + l, \tag{10–6}$$

And in this case, the number *n* cannot be a solution of the functional equation (10-4).

Proof. It is obvious that from formula (10-5) and from property (10-2) of additivity of the function *A*(*n*) on the digits of the number *n* formula (10-6) follows.

Proposition 10-4 is proved.

Let us combine the results of propositions (10-2) and (10-4) into a theorem.

Theorem 10-1. A number $n \in N$ is a solution of the functional equation (10-3) if and only if

$$v_5(n) = 0.$$

Thus, Theorem (10-1) completely describes all solutions of the functional equation (10-1) in the case *q=2*.

Similarly to the proof of theorem (10-1) the following theorem is proved

Theorem 10-2. A number $n \in N$ is a solution of the functional equation (10-1), where $q \in \{3, 4, 5, 6, 7, 8, 9\}$, if and only if the vertex of the corresponding order of the number n is zero

$$v_t(n) = 0.$$

q	3	4	5	6	7	8	9
t	4	3	2	2	2	2	2

Thus, theorems (10-1) and (10-2) completely describe all solutions of the functional equation (10-1) in the cases where $2 \le q \le 9$.

We now turn to the study of equation (10-1) in the cases when $q \ge 10$.

Proposition 10-5. Let

$$d(n) = k + 1, \quad n = \sum_{i=0}^{k} \alpha_i \cdot 10^i \text{ и } q = 10.$$

Then the equation

$$A(10 \cdot n) = 10 \cdot A(n) \tag{10–7}$$

has no solutions.

Proof.

$$A(10 \cdot n) = A\left(10 \cdot \sum_{L=0}^{k} \alpha_i \cdot 10^i\right) = A\left(\sum_{L=0}^{k} \alpha_i \cdot 10^{i+1}\right) =$$

$$= A\left(\alpha_0 \cdot 10 + \alpha_1 \cdot 10^2 + \cdots \alpha_k \cdot 10^{k+1}\right) = \alpha_0 \cdot R_1 + \alpha_1 \cdot R_2 + \cdots + \alpha_k \cdot R_{k+1}.$$

$$10 \cdot A(n) = 10 \cdot A\left(\sum_{L=0}^{k} \alpha_i \cdot 10^i\right) = 10 \cdot (\alpha_1 \cdot R_1 + \alpha_2 \cdot R_2 + \cdots + \alpha_k \cdot R_k).$$

Consider the difference

$$A(10 \cdot n) - 10 \cdot A(n) = \alpha_0 + \alpha_1(R_2 - 10R_1) + \cdots + \alpha_k(R_{k+1} - 10R_{k+1}).$$

It follows from the recurrence relation repunit (3-1) that

$$R_n - 10R_{n-1} = 1, \quad n = 2, 3, \ldots$$

Therefore.

$$A(10 \cdot n) - 10 \cdot A(n) = \alpha_0 + \alpha_1 + \cdots + \alpha_k = \sum_{i=o}^{k} \alpha_i > 0 \tag{10–9}$$

So, at $q = 10$ and for arbitrary $n \in N$ we obtain the inequality:

$$A(10n) > 10 \cdot A(n).$$

This inequality shows that equation (10-7) has no solutions.

Proposition 10-5 is proved. Similarly Proposition 10-5 is proved

Proposition 10-6. Let $n = \alpha_s \cdot 10^s, \quad \alpha_s \neq 0, \quad q \geq 10.$ Then

$$A(qn) > q \cdot A(n) \tag{10–10}$$

For further investigation it is necessary to remember the property of the function $A(n)$ в (4-3):

$$A(n_1 + n_2 + \cdots + n_S) \geq A(n_1) + A(n_2) + \cdots + A(n_s).$$

Now using this property of the function and the proposition (10-6) we prove the following theorem.

Theorem 10-3. Let $n \in N$ and $q \geq 10.$ Then the equation

$$A(qn) = q \cdot A(n)$$

has no solutions.

Proof. It is enough to show that the inequality

$$A(qn) > q \cdot A(n).$$

Let's

$$n = \sum_{i=0}^{k} \alpha_i \cdot 10^i, \quad \alpha_k \neq 0.$$

Next,

$$A(qn) = A\left(q \cdot \sum_{i=0}^{k} \alpha_i \cdot 10^i\right) = A\left(\sum_{i=0}^{k} (q\alpha_i) \cdot 10^i\right) \geq$$

$$\geq \sum_{i=0} A(q\alpha_i \cdot 10^i) > \sum_{i=0}^{k} q \cdot A(\alpha_i \cdot 10^i) = q \cdot \sum_{L=0}^{k} A(\alpha_i \cdot 10^i) = q \cdot A\left(\sum_{i=0}^{k} \alpha_i \cdot 10^i\right).$$

Thus, for arbitrary $n \in N$ and $q \geq 10.$ inequality is proved

$$A(qn) = q \cdot A(n)$$

This inequality proves the statement of Theorem (10-3).

So, theorems (10-1), (10-2) and (10-3) give the complete solution to the functional equation $A(qn) = qA(n)$ (10-1).

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А. ЖАРБОЛОВ

Республикалық физика-математика мектебі, Алматы, Қазақстан

ҰЛЫ КАПРЕКАР ЖОЛЫМЕН: А-ФУНКЦИЯ, РЕПЬЮНИТТЕР ЖӘНЕ ОЛАРДЫҢ ҚАСИЕТТЕРІ

Белгілі индиялық математик Д.Капрекарға сүйене отырып бұл еңбекте бүтін сандарды алудың жаңа әдісі көрсетілген: $A(n)=\frac{1}{9}\left(n-S(n)\right)$ *мұнда* $S(n)$ *– n санының ондық жүйедегі цифрларының қосындысы. А-функциясы натурал сандардың тамаша класы-*R_n *репьюниттер сандары жиынымен тығыз байланыста екені анықталды. Мақалада* R_n*-нің жаңа қасиеттері табылды. А-өзіндік туындаған сандар жиынының шексіз екендігі және әрбір а-туындаған санның дәл 10 генераторы бар екендігі дәлелденді. А-өзіндік туындаған сандардың таралуы жайындағы гипотеза негізделген және белгіленген.*

«Көрші» тұрған а-өзіндік туындаған сандар тізбегінің таралуы жайындағы гипотеза негізделген және белгіленген. «Көрші» тұрған а-өзіндік туындаған сандар тізбегінің формуласы анықталған. Капрекер функциясы $K(n)$ *және біз енгізген* $A(n)$ *функцияларының арасындағы еселік қатынастар зерттелді және толық шешілді.* $A(qn) = qA(n)$ *функционалдық теңдеуі зерттелді және толық шешілді.*

Түйін сөздер: *Д.Р.Капрекар, өзіндік туындаған сандар, репьюниттер, А -функция.*

ЖАРБОЛОВ АЛИХАН

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ПО ПУТИ ВЕЛИКОГО КАПРЕКАРА: А-ФУНКЦИЯ, РЕПЬЮНИТЫ И ИХ СВОЙСТВА

Следуя известному индийскому математику Д. Капрекару, в данной работе представлен новый способ получения целых чисел $A(n)=\frac{1}{9}\left(n-S(n)\right)$ *, где* $S(n)$ *– сумма цифр числа n в десятичной* *записи. Данная A-функция оказалась тесно связана с замечательным классом натуральных чисел – множеством чисел R_n репьюнит. Были найдены новые свойства R_n. Изучены свойства A-функции. Доказано, что множество a-самопорожденных чисел бесконечно и каждое a-порожденное число имеет ровно 10 генераторов. Обоснована и сформулирована гипотеза о распределении a-самопорожденных чисел. Обоснована и сформулирована гипотеза о распределении количества цепей «соседних» a-самопорожденных чисел и доказана полная согласованность двух гипотез. Найдены формулы для цепей «соседних» a-самопорожденных чисел. Изучены и полностью решены кратные отношения между функцией Капрекара $K(n)$ и введенной нами функцией $A(n)$. Исследованы и найдены все решения функционального уравнения $A(qn) = qA(n)$.*

Ключевые слова: *Д.Р. Капрекар, самопорожденные числа, репьюниты, A-функция.*