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DIRECT AND INVERSE PROBLEMS FOR A TWO-DIMENSIONAL PARABOLIC EQUATION WITH INVOLUTION

In this paper, using mappings of the involution type, we introduce a nonlocal analogue of the two-dimensional Laplace operator and consider the corresponding two-dimensional differential equation of parabolic type with involution. For this equation, the direct and inverse problems of finding the factors of the right-hand side, depending on the spatial variables, are studied.

The studied problems are solved by reducing them to direct and inverse problems for classical two-dimensional differential equations of parabolic type. On the basis of well-known theorems obtained for auxiliary problems, theorems on the existence and uniqueness of the solution of the studied problems are proved. The explicit form of solutions of the studied problems is constructed in the form of a series.

Key words: *involution, initial boundary value problem, nonlocal operator, inverse problem, parabolic equation.*

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ИНВОЛЮЦИЯСЫ БАР ЕКІ ӨЛШЕМДІ ПАРАБОЛАЛЫҚ ТЕҢДЕУ ҮШІН ТУРА ЖӘНЕ КЕРІ ЕСЕПТЕР

Бұл мақалада инволюциясы бар екі өлшемді параболалық теңдеу үшін тура және кері есептердің шешілу мүмкіндігін зерттеуге арналған. Қарастырылып отырған есептер параболалық типті классикалық екі өлшемді дифференциалдық теңдеулер үшін тура және кері есептер шығару арқылы шешіледі. Көмекші есептер бойынша алынған белгілі теоремалар негізінде қарастырылатын есептердің шешімдерінің бар және жалғыздығы туралы теоремалар дәлелденеді. Зерттелетін есептердің шешімдерінің айқын формасы қатар түрінде құрастырылады.

Түйін сөздер: *инволюция, бастапқы-шеттік есеп, локальды емес оператор, кері есеп, параболалық теңдеу.*

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ПРЯМЫЕ И ОБРАТНЫЕ ЗАДАЧИ ДЛЯ ДВУМЕРНОГО ПАРАБОЛИЧЕСКОГО УРАВНЕНИЯ С ИНВОЛЮЦИЕЙ

Статья посвящена исследованию вопросов разрешимости прямой и обратной задачи для двумерного параболического уравнения с инволюцией. Рассматриваемые задачи решаются сведением их к прямым и обратным задачам для классических двумерных дифференциальных уравнений параболического типа. На основании известных теорем, полученных относительно вспомогательных задач, доказаны теоремы о существования и единственности решения рассматриваемых задач. Явный вид решений исследуемых задач построен в виде ряда.

Ключевые слова: *инволюция, начально-краевая задача, нелокальный оператор, обратная задача, параболическое уравнение.*

1. Statement of the problem. Among differential equations with deviating arguments, a special place is occupied by equations with involutive deviations. Mapping S is called an involution if $S^2 = E$, where E is the identity mapping.

The theory of equations with involutively transformed arguments and their applications are described in detail in monographs [1–4]. To date, for differential equations with various types of involution, the well-posedness of boundary and initial-boundary value problems, the qualitative properties of solutions, and spectral questions have been well studied [5–16]. It is also necessary to note some recent works on inverse problems for heat equations and their fractional analogues [17–20].

The work is devoted to the study of solvability of direct and inverse problems for a two-dimensional parabolic equation with involution.

Let us consider the problem statement. Let $0 < p, q, T$ be real numbers, $\Pi = \left\{ x = (x_1, x_2) \in R^2 : 0 < x_1 < p, 0 < x_2 < q \right\}$ - a rectangle, $Q = (0,T) \times \Pi$.

For any point $x = (x_1, x_2) \in \Pi$ we consider the following mappings

$$S_0 x = (x_1, x_2), S_1 x = (p - x_1, x_2), S_2 x = (x_1, q - x_2), S_3 x = (p - x_1, q - x_2) .$$

It is obvious that for any $j = \overline{0,3}$ the equalities $S_j^2 x = x,$ are satisfied, i.e., mappings S_j are involutions. In addition, the equalities also hold:

$$S_1 \cdot S_2 = S_2 \cdot S_1 = S_3, S_1 \cdot S_3 = S_3 \cdot S_1 = S_2, S_2 \cdot S_3 = S_3 \cdot S_2 = S_1 .$$

Let a_j be real numbers, $j = \overline{0,3}$, Δ - the Laplace operator acting on the variables x_1 and x_2. For the function $v(x_1, x_2) \in C^2(\Pi)$ we can introduce the operator

$$Lv(x) \equiv a_0 \Delta v(S_0 x) + a_1 \Delta v(S_1 x) + a_1 \Delta v(S_2 x) + a_1 \Delta v(S_3 x) .$$

The operator L will be called a nonlocal Laplace operator. If $a_0=1, a_j=0, j=1,2,3$, L coincides with the ordinary two-dimensional Laplace operator.

Let us consider the following equation in the domain Q

$$u_t(t,x)-a_0\Delta u(t,S_0x)-a_1\Delta u(t,S_1x)-a_2\Delta u(t,S_2x)-a_3\Delta u(t,S_3x)=F(t,x),(t,x)\in Q . \quad (1)$$

Here $\Delta u(t,S_jx)$ means that $\Delta u(t,S_jx)=\Delta u(t,z)\big|_{z=S_jx}$, $j=\overline{0,3}$.

If $a_0=1, a_j=0, j=1,2,3$, equation (1) coincides with the classical parabolic equation.

Let us introduce the class of functions $W=\left\{u(t,x):u\in C\left(\bar{Q}\right)\cap C_{t,x}^{1,2}\left(Q\right)\right\}$ and consider the following problems in the domain Q .

Problem DP (Direct Problem). In the domain Q find the function $u(t,x)\in W$ satisfying equation (1) and the following conditions

$$u(0,x)=\varphi(x), x\in\bar{\Pi} , \quad (2)$$

$$u(t,0,x_2)=u(t,q,x_2)=0, 0\le x_2\le q, 0\le t\le T , \quad (3)$$

$$u(t,x_1,0)=u(t,x_1,p)=0, 0\le x_1\le p, 0\le t\le T . \quad (4)$$

Problem IP (Inverse Problem). Let $F(t,x)=g(t)f(x,y)$. Find the functions $u(t,x)\in W$ and $f(x)\in C\left(\bar{\Pi}\right)$ satisfying conditions (1) - (4) and the additional condition

$$u(t_0,x)=\psi(x), x\in\bar{\Pi} , \quad (5)$$

where t_0 is a fixed point in the segment $(0,T]$, $\varphi(x),\psi(x)$ and $g(t)$ are given functions.

It should be noted that considered here **DP** and **IP** problems for the case $a_0=1, a_j=0, j=1,2,3$, were studied in [21].

2. Auxiliary assertions. In this section, we present some well-known assertions proved in [26].

Let $a>0$. Consider the following equation in the domain Q:

$$w_t(t,x)-a^2\Delta w(t,x)=F(t,x),(t,x)\in Q . \quad (6)$$

Let us consider the following problems in the domain Q.

Problem 1. Find the function $w(t,x)\in W$, which in the domain Q satisfies equation (6) and the conditions

$$w(0,x)=\varphi(x), x\in\bar{\Pi} , \quad (7)$$

$$w(t,0,x_2)=w(t,q,x_2)=0, 0\le x_2\le q, 0\le t\le T, \quad (8)$$

$$w(t,x_1,0)=w(t,x_1,p)=0, 0\le x_1\le p, 0\le t\le T . \quad (9)$$

Problem 2. Let $F(t,x)=g(t)f(x,y)$. Find the functions $w(t,x)\in W$ and $f(x)\in C\left(\bar{\Pi}\right)$, which in the domain Q satisfy conditions (6)-(9) and the additional condition

$$w(t_0, x) = \psi(x), x \in \bar{\Pi} \ ,$$

where t_0 is a fixed point in the segment $(0,T]$, $\varphi(x), \psi(x)$ and $g(t)$ are given functions.

In [26], the following assertions were proved for problems 1 and 2.

Theorem 1. If there is a solution to Problem 1 satisfying the conditions

$$\lim_{x_1 \to 0+} u_{x_1}(t, x_1, x_2) \sin \frac{\pi m x_1}{p} = \lim_{x_1 \to p-} u_{x_1}(t, x_1, x_2) \sin \frac{\pi n x_1}{p} = 0, 0 \le x_2 \le q, 0 \le t \le T,$$

$$\lim_{x_2 \to 0+} u_{x_2}(t, x_1, x_2) \sin \frac{\pi n x_2}{q} = \lim_{x_2 \to q-} u_{x_1}(t, x_1, x_2) \sin \frac{\pi n x_2}{q} = 0, 0 \le x_1 \le p, 0 \le t \le T,$$

then it is unique.

Theorem 2. If $\varphi(x) \in C^2\left(\bar{\Pi}\right), F(t,x) \in C^{0,2}_{t,x}\left(\bar{Q}\right)$ and conditions

$$\varphi(0, x_2) = \varphi(p, x_2) = 0, 0 \le x_2 \le q, \varphi(x_1, 0) = \varphi(x_1, q) = 0, 0 \le x_1 \le p \ , \tag{10}$$

$$\begin{aligned} &F(t,0,x_2) = F(t,p,x_2) = 0, 0 \le x_2 \le q, 0 \le t \le T, \\ &F(t,x_1,0) = F(t,x_1,q) = 0, 0 \le x_1 \le p, 0 \le t \le T \end{aligned} \ , \tag{11}$$

are satisfied, then the solution to Problem 1 from the class W exists, is unique and is represented as a series

$$w(t,x) = \sum_{m,n=1}^{\infty} w_{mn}(t) X_{mn}(x), \tag{12}$$

where

$$X_{mn}(x) = \frac{2}{\sqrt{pq}} \sin \mu_m x_1 \sin \nu_n x_2, \mu_m = \frac{\pi m}{p}, \nu_n = \frac{\pi n}{q}, \mu_{mn}^2 = \mu_m^2 + \nu_n^2 \ , \tag{13}$$

$$w_{mn}(t) = \varphi_{mn} e^{-\mu_{mn}^2 a^2 t} + \int_0^t F_{mn}(s)\, e^{-\mu_{mn}^2 a^2 (t-s)} ds \ ,$$

$$F_{mn}(t) = (F, X_{mn}) \equiv \int_{\Pi} F(t,x) X_{mn}(x) dx, \varphi_{mn} = (\varphi, X_{mn}) \equiv \int_{\Pi} \varphi(x) X_{mn}(x) dx \ .$$

Theorem 3. Let the function $\varphi(x)$ belong to the class $C^4\left(\bar{\Pi}\right)$ and satisfy the conditions

$$\begin{aligned} &\varphi(0,x_2) = \varphi_{x_1 x_1}(0,x_2) = \varphi(p,x_2) = \varphi_{x_1 x_1}(p,x_2), 0 \le x_2 \le q, \\ &\varphi(x_1,0) = \varphi_{x_2 x_2}(x_1,0) = \varphi(x_1,q) = \varphi_{x_2 x_2}(x_1,q), 0 \le x_1 \le p \end{aligned} \ , \tag{14}$$

and the function $F(t, x)$ satisfy the conditions of Theorem 2. Then the solution to Problem 1 exists, is unique, represented in the form of series (12) and belongs to the class $C^1\left(\bar{Q}\right) \cap C^2_x(Q)$.

Now we will consider the main assertions for Problem 2.

Theorem 4. Let $g(t)=1$. If functions $\varphi(x)$ and $\psi(x)$ satisfy conditions (14), then the solution to Problem 2 exists, is unique, and can be represented in the form of series

$$w(t,x)=\sum_{m,n=1}^{\infty} w_{mn}(t)X_{mn}(x)\ , \tag{15}$$

$$f(x)=\sum_{m,n=1}^{\infty} f_{mn}(t)X_{mn}(x)\ , \tag{16}$$

where

$$f_{mn}=\left(a\mu_{mn}\right)^2\left(\frac{\psi_{mn}}{1-e^{-(a\mu_{mn})^2 t_0}}-\frac{e^{-(a\mu_{mn})^2 t_0}\varphi_{mn}}{1-e^{-(a\mu_{mn})^2 t_0}}\right),$$

$$w_{mn}(t)=\frac{e^{-(a\mu_{mn})^2 t}-e^{-(a\mu_{mn})^2 t_0}}{1-e^{-(a\mu_{mn})^2 t_0}}\varphi_{mn}+\frac{1-e^{-(a\mu_{mn})^2 t}}{1-e^{-(a\mu_{mn})^2 t_0}}\psi_{mn}\ ,$$

$$\varphi_{mn}=(\varphi,X_{mn}),\psi_{mn}=(\psi,X_{mn})\ .$$

Theorem 5. Let $g(t)\neq 1, g(t)\in C[0,T]$ and $|g(t)|\geq g_0>0$. If the functions $\varphi(x)$ and $\psi(x)$ satisfy conditions (14), the solution to Problem 2 exists, is unique, and can be represented in the form of series (15) and (16). In this case, the coefficients of these series are determined by the equalities

$$f_{mn}=\frac{1}{g_{mn}(t_0)}\left(\psi_{mn}-\varphi_{mn}e^{-(a\mu_{mn})^2 t_0}\right),$$

$$w_{mn}(t)=\left(1-\frac{g_{mn}(t)}{g_{mn}(t_0)}\right)e^{-(a\mu_{mn})^2 t}\varphi_{mn}+\frac{g_{mn}(t)}{g_{mn}(t_0)}\psi_{mn}\ ,$$

$$g_{mn}(t)=\int_0^t g(s)e^{-(a\mu_{mn})^2(t-s)}ds\ .$$

3. Investigation of the direct problem. In this section, we consider the direct problem.

Let the function $u(t, x)$ be a solution to equation (1). Changing the point (t, x) by $(t,S_j x), j=1,2,3$ in equation (1) for the function $u(t, x)$, we obtain the following system

$$U_t=A\Delta U+\bar{F} \tag{17}$$

where

$$U=\begin{pmatrix} u(t,x)\\ u(t,S_1x)\\ u(t,S_2x)\\ u(t,S_3x)\end{pmatrix}, A=\begin{pmatrix} a_0 & a_1 & a_2 & a_3\\ a_1 & a_0 & a_3 & a_2\\ a_2 & a_3 & a_0 & a_1\\ a_3 & a_2 & a_1 & a_0\end{pmatrix}, \bar{F}=\begin{pmatrix} F(t,x)\\ F(t,S_1x)\\ F(t,S_2x)\\ F(t,S_3x)\end{pmatrix}.$$

It is not difficult to show that the eigenvectors and the corresponding eigenvalues of the matrix A can be written as:

$$V_1=(1,1,1,1)^T, V_2=(1,1,-1,-1)^T, V_3=(1,-1,1,-1)^T, V_4=(1,-1,-1,1)^T,$$

$$\varepsilon_1=a_0+a_1+a_2+a_3; \varepsilon_2=a_0+a_1-a_2-a_3; \varepsilon_3=a_0-a_1+a_2-a_3; \varepsilon_4=a_0-a_1-a_2+a_3.$$

Let us multiply equation (17) scalarly to the vector $V_j, j=1,2,3,4$. Then, taking into account the symmetry of the matrix A and the equality, $AV_j=\varepsilon_j V_j$ we get

$$\left(U_t,V_j\right)=\Delta\left(U,AV_j\right)+\left(\bar{F},V_j\right)\Rightarrow\left(U_t,V_j\right)=\varepsilon_j\Delta\left(U,V_j\right)+\left(\bar{F},V_j\right).$$

Further, we introduce the notation $w_j(t,x)=\left(U,V_j\right), j=1,2,3,4$. Then for the function $w_j(t,x)$ we obtain a system of equations

$$\frac{\partial w_j(t,x)}{\partial t}=\varepsilon_j\Delta w_j(t,x)+\tilde{F}_j(t,x),(t,x)\in Q, j=1,2,3,4.$$

where

$$\tilde{F}_j(t,x)=\nu_{1j}F(t,x)+\nu_{2j}F(t,S_1x)+\nu_{3j}F(t,S_2x)+\nu_{4j}F(t,S_3x).$$

In addition, from boundary conditions (2) and (3) we obtain

$$w_i(0,x)=\sum_{j=0}^{3}v_{i,j+1}u\left(0,S_jx\right)=\sum_{j=0}^{3}v_{i,j+1}\varphi\left(S_jx\right)\equiv\tilde{\varphi}_i(x), i=\overline{1,4},$$

$$\begin{gathered}
w_j(t,0,x_2)=\nu_{1j}u(t,0,x_2)+\nu_{2j}u(t,p,x_2)+\nu_{3j}u(t,0,q-x_2)+\nu_{4j}u(t,p,q-x_2)=0\\
w_j(t,p,x_2)=\nu_{1j}u(t,p,x_2)+\nu_{2j}u(t,0,x_2)+\nu_{3j}u(t,p,q-x_2)+\nu_{4j}u(t,0,q-x_2)=0\\
w_j(t,x_1,0)=\nu_{1j}u(t,x_1,0)+\nu_{2j}u(t,p-x_1,0)+\nu_{3j}u(t,x_1,q)+\nu_{4j}u(t,p-x_1,q)=0\\
w_j(t,x_1,q)=\nu_{1j}u(t,x_1,q)+\nu_{2j}u(t,p-x_1,q)+\nu_{3j}u(t,x_1,0)+\nu_{4j}u(t,p-x_1,0)=0
\end{gathered}.$$

Thus, we have proved the following assertion.

Theorem 6. Let $V_j=(v_{1,j},v_{2,j},v_{3,j},v_{4,j})^T$ be an eigenvector of the matrix A and ε_j – the corresponding eigenvalue and $\varepsilon_j\neq 0, j=1,2,3,4$. If the function $u(t, x)$ is a solution to the DP problem, then the functions

$$w_j(t,x)=\nu_{1j}u(t,x)+\nu_{2j}u(t,S_1x)+\nu_{3j}u(t,S_2x)+\nu_{4j}u(t,S_3x), j=1,2,3,4$$

are solutions to the following problems

$$\frac{\partial w_j(t,x)}{\partial t}=\varepsilon_j\Delta w_j(t,x)+\tilde{F}_j(t,x),(t,x)\in Q \tag{18j}$$

$$w_j(0,x)=\tilde{\varphi}_j(x), x\in\bar{\Pi}, \tag{19j}$$

$$\begin{aligned} & w_j(t,0,x_2) = w_j(t,p,x_2) = 0, 0 \le t \le T, 0 \le x_2 \le q; \\ & w_j(t,x_1,0) = w_j(t,x_1,q) = 0, 0 \le t \le T, 0 \le x_1 \le p \end{aligned}, \tag{20j}$$

where functions $\tilde{F}_j(t,x)$ and $\tilde{\varphi}_i(x)$ are defined by the equalities

$$\tilde{F}_j(t,x) = \sum_{i=0}^{3} v_{i+1,j} F(t, S_i x), \tilde{\varphi}_j(x) = \sum_{i=0}^{3} v_{i+1,j} \varphi(S_i x), \; j = \overline{1,4}\;. \tag{21j}$$

Let us prove the converse assertion. Note that the system of algebraic equations $w_j(t,x) = (U, V_j),\; j = 1,2,3,4$ can be rewritten as $W = VU$, where

$$W = (w_1(t,x), w_2(t,x), w_3(t,x), w_4(t,x))^T, U = (u(t,x), u(t,S_1x), u(t,S_2x), u(t,S_3x))^T,$$

$$V = \begin{pmatrix} v_{1,1} & v_{1,2} & v_{1,3} & v_{1,4} \\ v_{2,1} & v_{2,2} & v_{2,3} & v_{2,4} \\ v_{3,1} & v_{3,2} & v_{3,3} & v_{3,4} \\ v_{4,1} & v_{4,2} & v_{4,3} & v_{4,4} \end{pmatrix} \equiv \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

It is easy to show that $V^{-1} = \frac{1}{4}V$. Hence, $W = VU \Rightarrow U = \frac{1}{4}VW$, which is the same as

$$\begin{aligned} u(t,x) &= \frac{1}{4}\left[w_1(t,x) + w_2(t,x) + w_3(t,x) + w_4(t,x)\right], \\ u(t,S_1x) &= \frac{1}{4}\left[w_1(t,x) + w_2(t,x) - w_3(t,x) - w_4(t,x)\right], \\ u(t,S_2x) &= \frac{1}{4}\left[w_1(t,x) + w_2(t,x) - w_3(t,x) - w_4(t,x)\right], \\ u(t,S_3x) &= \frac{1}{4}\left[w_1(t,x) + w_2(t,x) - w_3(t,x) - w_4(t,x)\right]. \end{aligned}$$

The following assertion is valid.

Theorem 7. Let $V_j = (v_{1,j}, v_{2,j}, v_{3,j}, v_{4,j})^T$ be an eigenvector of the matrix A and ε_j – the corresponding eigenvalue and $\varepsilon_j \ne 0,\; j = 1,2,3,4$. If functions $\tilde{F}_j(t,x)$ and $\tilde{\varphi}_j(x)$ are defined by equalities (21j) and $w_j(t,x)$ are solutions to problems (18j) - (20j), j = 1,2,3,4, then the function

$$u(t,x) = \frac{1}{4}\left[w_1(t,x) + w_2(t,x) + w_3(t,x) + w_4(t,x)\right] \tag{22}$$

is a solution to the DP problem.

Proof. Let the functions $w_j(t,x)$ be solutions to problems (18j) - (20j), j = 1,2,3,4. Let us construct a vector $U = \frac{1}{4}VW$ based on these functions. Then

$$U_t = \frac{1}{4} V W_t, \Delta U = \frac{1}{4} V \Delta W, A\Delta U = \frac{1}{4} AV\Delta W .$$

Hence

$$U_t - A\Delta U = \frac{1}{4} V W_t - \frac{1}{4} AV\Delta W .$$

As $AV = \varepsilon V$ и $W_t - \varepsilon \Delta W = \tilde{F}$, where $\tilde{F} = \left(\tilde{F}_1(t,x), \tilde{F}_2(t,x), \tilde{F}_3(t,x), \tilde{F}_4(t,x)\right)^T$, then

$$U_t - A\Delta U = \frac{1}{4} V\left(W_t - \Delta W\right) = \frac{1}{4} V\tilde{F} .$$

As $\tilde{F} = V\bar{F}$ и $\frac{1}{4} VV = E$, then $\frac{1}{4} V\tilde{F} = \bar{F}$ and thus

$$U_t - A\Delta U = \bar{F} .$$

Hence, in particular, for the function $u(t,x)$ in (22) we get

$$u_t(t,x) - a_0 \Delta u(t,x) - a_1 \Delta u(t,S_1x) - a_2 \Delta u(t,S_2x) - a_3 \Delta u(t,S_3x) = f(t,x) ,$$

i.e., the function $u(t,x)$ satisfies equation (1). Further, as the functions $w_j(t,x),\ j = 1,2,3,4$ satisfy conditions (20j), it is obvious that the function $u(t,x)$ satisfies conditions (3) and (4). And finally, if we denote vectors $\tilde{\Phi} = \left(\tilde{\varphi}_1(x), \tilde{\varphi}_2(x), \tilde{\varphi}_3(x), \tilde{\varphi}_4(x)\right)^T$ and $\Phi = \left(\varphi(x), \varphi(S_1x), \varphi(S_2x), \varphi(S_3x)\right)^T$ through $\tilde{\Phi}$ and Φ, then due to equality $\tilde{\Phi} = V\Phi$ for $U = \frac{1}{4} VW$ we get

$$U\big|_{t=0} = \frac{1}{4} V W\big|_{t=0} = \frac{1}{4} V\left(V\Phi\right) = \Phi .$$

Hence, for the function $u(t,x)$ in (22) we obtain the condition $u(0,x) = \varphi(x)$. Thus, the function $u(t,x)$ in (22) satisfies all conditions of the DP problem. The theorem is proved.

It follows from this theorem that to find a solution to the DP problem, it is sufficient to solve problems (18j) - (20j).

The following assertion is valid.

Theorem 8. Let $0 < \varepsilon_j,\ j = 1,2,3,4,$ $\varphi(x) \in C^2\left(\bar{\Pi}\right), F(t,x) \in C^{0,2}_{t,x}\left(\bar{Q}\right)$ and conditions (10) and (11) are satisfied. Then the solution to the **DP** problem exists, is unique, and is represented as a series

$$u(t,x) = \sum_{m,n=1}^{\infty} w_{(2m-1)(2n-1)}(t,\varepsilon_1) X_{(2m-1)(2n-1)}(x) + \sum_{m,n=1}^{\infty} w_{(2m-1)2n}(t,\varepsilon_2) X_{(2m-1)2n}(x)$$

$$+\sum_{m,n=1}^{\infty} w_{2m(2n-1)}(t,\varepsilon_3)X_{2m(2n-1)}(x)+\sum_{m,n=1}^{\infty} w_{2m2n}(t,\varepsilon_4)X_{2m2n}(x)\ . \qquad (23)$$

where

$$w_{mn}(t,\varepsilon_j)=\varphi_{mn}e^{-\mu_{mn}^2\varepsilon_j t}+\int_0^t F_{mn}(s)\,e^{-\mu_{mn}^2\varepsilon_j(t-s)}ds\ ,$$

$$F_{mn}(t)=(F,X_{mn})\equiv\int_\Pi F(t,x)X_{mn}(x)dx,\varphi_{mn}=(\varphi,X_{mn})\equiv\int_\Pi \varphi(x)X_{mn}(x)dx\ ,$$

and $X_{mn}(x),\mu_m,\nu_n$ are determined from (13)

Proof. Let $\varphi(x)\in C^2\left(\overline{\Pi}\right),F(t,x)\in C_{t,x}^{0,2}\left(\overline{Q}\right)$ and for them conditions (10) and (11) are satisfied. Let us introduce the functions

$$\tilde{\varphi}_j(x)=\sum_{i=0}^{3} v_{j,i+1}\varphi\left(S_i x\right),\tilde{F}_j(t,x)=\sum_{i=0}^{3} v_{j,i+1}F\left(t,S_i x\right),\ j=\overline{1,4}\ .$$

and consider problems (18_j) - (20_j), j = 1,2,3,4.

It is obvious that for all $j=\overline{1,4}$ the inclusions $\tilde{F}_j(t,x)\in C_{t,x}^{0,2}\left(\overline{Q}\right),\tilde{\varphi}_i(x)\in C^2\left(\overline{\Pi}\right)$ and conditions

$$\begin{aligned}
&\tilde{\varphi}_j(0,x_2)=v_{j,1}\varphi(0,x_2)+v_{j,2}\varphi(p,x_2)+v_{j,3}\varphi(0,q-x_2)+v_{j,4}\varphi(p,q-x_2)=0,\\
&\tilde{\varphi}_j(p,x_2)=v_{j,1}\varphi(p,x_2)+v_{j,2}\varphi(0,x_2)+v_{j,3}\varphi(p,q-x_2)+v_{j,4}\varphi(0,q-x_2)=0,\\
&\tilde{\varphi}_j(x_1,0)=v_{j,1}\varphi(x_1,0)+v_{j,2}\varphi(p-x_1,0)+v_{j,3}\varphi(x_1,q)+v_{j,4}\varphi(p-x_1,q)=0,\\
&\tilde{\varphi}_j(x_1,q)=v_{j,1}\varphi(x_1,q)+v_{j,2}\varphi(p-x_1,q)+v_{j,3}\varphi(x_1,0)+v_{j,4}\varphi(p-x_1,0)=0
\end{aligned}\ .$$

are satisfied.

It is shown similarly that the function $\tilde{F}_j(t,x)$ satisfies the conditions

$$\begin{aligned}
&\tilde{F}_j(t,0,x_2)=\tilde{F}_j(t,p,x_2)=0,0\le x_2\le q,0\le t\le T,\\
&\tilde{F}_j(t,x_1,0)=\tilde{F}_j(t,x_1,q)=0,0\le x_1\le p,0\le t\le T
\end{aligned}\ .$$

Hence, the functions $\tilde{\varphi}_j(x)$ and $\tilde{F}_j(t,x)$ satisfy all conditions of Theorem 2. Then, by the assertion of this theorem, the solution to problems (18_j) - (20_j), j = 1,2,3,4, from the class W exists, is unique, and can be represented as a series

$$w_j(t,x)=\sum_{m,n=1}^{\infty} w_{mn}^j(t)X_{mn}(x),\ j=1,2,3,4\ , \qquad (24)$$

where

$$w_{mn}^j(t)=\varphi_{mn}^j e^{-\mu_{mn}^2\varepsilon_j t}+\int_0^t F_{mn}^j(s)\,e^{-\mu_{mn}^2\varepsilon_j(t-s)}ds\ ,$$

$$F_{mn}^{j}(t)=(\tilde{F}_j,X_{mn})\equiv\int_{\Pi}\tilde{F}_j(t,x)X_{mn}(x)dx,\varphi_{mn}^{j}=(\tilde{\varphi}_j,X_{mn})\equiv\int_{\Pi}\tilde{\varphi}_j(x)X_{mn}(x)dx .$$

Further, the function $\tilde{z}(x)=z(S_jx)$ satisfies the equality

$$\int_{\Pi}\tilde{z}(x)X_{mn}(x)dx=\int_{\Pi}z(S_jx)X_{mn}(x)dx=\int_{\Pi}z(x)X_{mn}(S_jx)dx .$$

As

$$X_{mn}(S_1x)=X_{mn}(p-x_1,x_2)=\frac{2}{\sqrt{pq}}\sin\mu_m(p-x_1)\sin\nu_nx_2=(-1)^{m+1}\frac{2}{\sqrt{pq}}\sin\mu_mx_1\sin\nu_nx_2=(-1)^{m+1}X_{mn}(x)$$

$$X_{mn}(S_2x)=X_{mn}(x_1,q-x_2)=\frac{2}{\sqrt{pq}}\sin\mu_mx_1\sin\nu_n(q-x_2)=(-1)^{n+1}\frac{2}{\sqrt{pq}}\sin\mu_mx_1\sin\nu_nx_2=(-1)^{n+1}X_{mn}(x)$$

$$X_{mn}(S_3x)=X_{mn}(p-x_1,q-x_2)=(-1)^{m+n}\frac{2}{\sqrt{pq}}\sin\mu_mx_1\sin\nu_nx_2=(-1)^{m+n}X_{mn}(x)$$

then for coefficients φ_{mn}^{j} we get

$$\varphi_{mn}^{j}=(\tilde{\varphi}_j,X_{mn})\equiv\int_{\Pi}\left(\sum_{i=0}^{3}v_{j,i+1}\varphi(S_ix)\right)X_{mn}(x)dx=\sum_{i=0}^{3}v_{j,i+1}\int_{\Pi}\varphi(S_ix)X_{mn}(x)dx$$

$$=\sum_{i=0}^{3}v_{j,i+1}\int_{\Pi}\varphi(x)X_{mn}(S_ix)dx=\left(v_{j,1}+(-1)^{m+1}v_{j,2}+(-1)^{n+1}v_{j,3}+(-1)^{m+n}v_{j,4}\right)\int_{\Pi}\varphi(x)X_{mn}(x)dx$$

Further, using the values $\nu_{ji}, i,j=1,2,3,4$, for all $m,n=1,2,...$, we get,

$$\varphi_{mn}^{1}=\left(1+(-1)^{m+1}+(-1)^{n+1}+(-1)^{m+n}\right)\varphi_{mn}=\begin{cases}4\varphi_{(2k+1)(2l+1)}, m=2k+1, n=2l+1\\0, other\ m,n\end{cases}, \quad (25)$$

$$\varphi_{mn}^{2}=\left(1+(-1)^{m+1}-(-1)^{n+1}-(-1)^{m+n}\right)\varphi_{mn}=\begin{cases}4\varphi_{(2k+1)2l}, m=2k+1, n=2l,\\0, other\ m,n\end{cases}, \quad (26)$$

$$\varphi_{mn}^{3}=\left(1-(-1)^{m+1}+(-1)^{n+1}-(-1)^{m+n}\right)\varphi_{mn}=\begin{cases}4\varphi_{2k(2l+1)}, m=2k, n=2l+1,\\0, other\ m,n\end{cases}, \quad (27)$$

$$\varphi_{mn}^{4}=\left(1-(-1)^{m+1}-(-1)^{n+1}+(-1)^{m+n}\right)\varphi_{mn}=\begin{cases}4\varphi_{2k2l}, m=2k, n=2l,\\0, other\ m,n\end{cases}. \quad (28)$$

The coefficients $F_{mn}^{j}(t)$ satisfy similar conditions. Then, series (24) will be written as

$$w_1(t,x)=4\sum_{m,n=1}^{\infty}w_{(2m-1)(2n-1)}(t,\varepsilon_1)X_{(2m-1)(2n-1)}(x) ,$$

$$w_2(t,x)=4\sum_{m,n=1}^{\infty}w_{(2m+1)2n}(t,\varepsilon_2)X_{(2m+1)2n}(x) ,$$

$$w_3(t,x) = 4\sum_{m,n=1}^{\infty} w_{2m(2n-1)}(t,\varepsilon_3) X_{2m(2n+1)}(x) \;,$$

$$w_4(t,x) = 4\sum_{m,n=1}^{\infty} w_{2m2n}(t,\varepsilon_4) X_{2m2n}(x) \;,$$

where

$$w_{mn}(t,\varepsilon_j) = \varphi_{mn} e^{-\mu_{mn}^2 \varepsilon_j t} + \int_0^t F_{mn}\; e^{-\mu_{mn}^2 \varepsilon_j (t-s)} ds$$

Substituting the obtained expressions into equality (22) for the solution of the DP problem, we obtain the representation (23). The theorem is proved.

4. Investigation of the inverse problem.

In this section, we consider the inverse problem.

Let $g(t) = 1$, and functions $\varphi(x)$ and $\psi(x)$ belong to the class $C^4\left(\bar{\Pi}\right)$ and satisfy conditions (14). Using these functions, we construct $\tilde{\varphi}_j(x), \tilde{\psi}_j(x)$:

$$\tilde{\varphi}_j(x) = \nu_{1j}\varphi(x) + \nu_{2j}\varphi(S_1x) + \nu_{3j}\varphi(S_2x) + \nu_{4j}\varphi(S_3x) \;, \tag{29}$$

$$\tilde{\psi}_j(x) = \nu_{1j}\psi(x) + \nu_{2j}\psi(S_1x) + \nu_{3j}\psi(S_2x) + \nu_{4j}\psi(S_3x),\; j = 1,2,3,4 \;. \tag{30}$$

For the unknown function $f(x)$, we construct the function

$$\tilde{f}_j(x) = \nu_{1j} f(x) + \nu_{2j} f(S_1x) + \nu_{3j} f(S_2x) + \nu_{4j} f(S_3x),\; j = 1,2,3,4$$

and consider the problem of determining a pair of functions $\left\{w_j(t,x),\, \tilde{f}_j(x)\right\}$ satisfying the conditions

$$\frac{\partial w_j(t,x)}{\partial t} = \varepsilon_j \Delta w_j(t,x) + \tilde{f}(x), (t,x) \in Q \tag{31j}$$

$$w_j(0,x) = \tilde{\varphi}_j(x), x \in \bar{\Pi}, w_j(T,x) = \tilde{\psi}_j(x), x \in \bar{\Pi} \;, \tag{32j}$$

$$\begin{aligned} &w_j(t,0,x_2) = w_j(t,p,x_2) = 0, 0 \le t \le T, 0 \le x_2 \le q; \\ &w_j(t,x_1,0) = w_j(t,x_1,q) = 0, 0 \le t \le T, 0 \le x_1 \le p \end{aligned} \;. \tag{33j}$$

Note that if a function $z(x)$ belongs to the class $C^4\left(\bar{\Pi}\right)$ and satisfies conditions (14), then the functions

$$\tilde{z}(x) = \nu_{1j}z(x) + \nu_{2j}z(S_1x) + \nu_{3j}z(S_2x) + \nu_{4j}z(S_3x),\; j = 1,2,3,4$$

also belong to the class $C^4\left(\bar{\Pi}\right)$ and by virtue of the equalities

$$z_{x_1x_1}\left(x_1,x_2\right) = z_{x_1x_1}\left(p - x_1, x_2\right), z_{x_2x_2}\left(x_1,x_2\right) = z_{x_1x_1}\left(x_1, q - x_2\right)$$

satisfy conditions (14).

Therefore, the functions $\tilde{\varphi}_j(x), \tilde{\psi}_j(x)$ belong to the class $C^4\left(\overline{\Pi}\right)$ and satisfy conditions (14).

Therefore, by the assertion of Theorem 4, the solution to problem (31j) - (33j) exists, is unique and is represented in the form

$$w_j(t,x) = \sum_{m,n=1}^{\infty} w_{mn}^{(j)}(t) X_{mn}(x) , \tag{34}$$

$$f_j(x) = \sum_{m,n=1}^{\infty} f_{mn}^{(j)}(t) X_{mn}(x) , \tag{35}$$

where

$$f_{mn}^{(j)} = \mu_{mn}^2 \varepsilon_j \left(\frac{\psi_{mn}^{(j)}}{1 - e^{-\mu_{mn}^2 \varepsilon_j t_0}} - \frac{e^{-(\varepsilon_j \mu_{mn})^2 t_0} \varphi_{mn}^{(j)}}{1 - e^{-\mu_{mn}^2 \varepsilon_j t_0}} \right), \tag{36}$$

$$w_{mn}^{(j)}(t) = \frac{e^{-\mu_{mn}^2 \varepsilon_j t} - e^{-\mu_{mn}^2 \varepsilon_j t_0}}{1 - e^{-\mu_{mn}^2 \varepsilon_j t_0}} \varphi_{mn}^{(j)} + \frac{1 - e^{-\mu_{mn}^2 \varepsilon_j t}}{1 - e^{-\mu_{mn}^2 \varepsilon_j t_0}} \psi_{mn}^{(j)} , \tag{37}$$

$$\varphi_{mn}^{(j)} = (\varphi_j, X_{mn}), \psi_{mn}^{(j)} = (\psi_j, X_{mn}) .$$

Let us show that the pair of functions $\{u(t,x), f(x)\}$ defined by the equalities

$$u(t,x) = \frac{1}{4}\left[w_1(t,x) + w_2(t,x) w_3(t,x) + w_4(t,x) \right] , \tag{38}$$

$$f(x) = \frac{1}{4}\left[\tilde{f}_1(x) + \tilde{f}_2(x) + \tilde{f}_3(x) + \tilde{f}_4(x) \right] \tag{39}$$

will be a solution to the IP problem in the case $g(t) = 1$.

Indeed, if functions $w_j(t,x)$ and $\tilde{f}_j(x)$ are solutions to problems (31j) - (33j), then for vectors

$$W = \left(w_1(t,x), w_2(t,x), w_3(t,x), w_4(t,x) \right)^T , \tilde{f} = \left(\tilde{f}_1(x), \tilde{f}_2(x), \tilde{f}_3(x), \tilde{f}_4(x) \right)^T$$

the equality $W_t - \varepsilon \Delta W = \tilde{f}$ is satisfied. Consider the vectors $U = \frac{1}{4} VW, \overline{f} = \frac{1}{4} V\tilde{f}$. Then

$$U_t - A\Delta U = \frac{1}{4} VW_t - \frac{1}{4} AV\Delta W = \frac{1}{4} VW_t - \frac{1}{4} \varepsilon V \Delta W = \frac{1}{4} V \left(W_t - \varepsilon \Delta W \right) = \frac{1}{4} V\tilde{f} = \overline{f}$$

If we introduce the notation $F = \left(f(x), f(S_1 x), f(S_2 x), f(S_3 x) \right)^T$, then from equalities

$$\tilde{f}_j(x) = \nu_{1j} f(x) + \nu_{2j} f(S_1 x) + \nu_{3j} f(S_2 x) + \nu_{4j} f(S_3 x), \ j = 1,2,3,4 \ ,$$

it follows that $\tilde{f} = VF$. Therefore, $\frac{1}{4} V\tilde{f} = \frac{1}{4} VVF = F$, i.e., $\overline{f} = F$. Hence,

$$U_t - A\Delta U = F \ .$$

In particular, for functions $u(t, x)$ and $f(x)$ we obtain the equality

$$u_t(t,x) - a_0\Delta u(t,S_0x) - a_1\Delta u(t,S_1x) - a_2\Delta u(t,S_2x) - a_3\Delta u(t,S_3x) = f(x), (t,x)\in Q \ ,$$

i.e., the pair of functions $\{u(t,x), f(x)\}$ satisfies equation (1) in the case $g(t)=1$.

That conditions (2) and (3) are satisfied is checked in the same way as in the case of the direct problem. Indeed, for t = 0 we have $U\big|_{t=0} = \frac{1}{4}VW\Big|_{t=0} = \frac{1}{4}V\tilde{\Phi},$ where $\tilde{\Phi}$ is used to denote the vector $\tilde{\Phi} = \left(\tilde{\varphi}_1(x), \tilde{\varphi}_2(x), \tilde{\varphi}_3(x), \tilde{\varphi}_4(x)\right)^T$. The structure of functions $\tilde{\varphi}_j(x)$ is determined by equalities (29) and, therefore, the vector equality $\tilde{\Phi} = V\Phi$ holds where $\Phi = \left(\varphi(x), \varphi(S_1x), \varphi(S_2x), \varphi(S_3x)\right)^T$. Therefore, $\frac{1}{4}V\tilde{\Phi} = \frac{1}{4}VV\Phi = \Phi$. Hence, for functions $u(t, x)$ we get the equality $u(0,x) = \varphi(x)$. Fulfillment of the condition $u(T,x) = \psi(x)$ is proved in a similar way. And finally, due to the fulfillment of conditions (33j), the function $u(t, x)$ from equality (38) also satisfies the boundary conditions (3).

Now, we can find the explicit form of the functions $u(t, x)$ and $f(x)$. To do this, we substitute expressions from (34) - (37) into the right-hand side of equalities (38) and (39).

For coefficients $\varphi_{mn}^{(j)} = (\varphi_j, X_{mn})$ equalities (25)–(28) are satisfied. Similar equalities are also valid for coefficients $\psi_{mn}^{(j)} = (\psi_j, X_{mn})$. Then

$$w_1(t,x) = 4\sum_{m,n=1}^{\infty} w_{(2m-1)(2n-1)}(t,\varepsilon_1)X_{(2m-1)(2n-1)}(x) \ ,$$

$$w_2(t,x) = 4\sum_{m,n=1}^{\infty} w_{(2m-1)2n}(t,\varepsilon_2)X_{(2m-1)2n}(x) \ ,$$

$$w_3(t,x) = 4\sum_{m,n=1}^{\infty} w_{2m(2n-1)}(t,\varepsilon_3)X_{2m(2n-1)}(x) \ ,$$

$$w_4(t,x) = 4\sum_{m,n=1}^{\infty} w_{2m2n}(t,\varepsilon_4)X_{2m2n}(x) \ ,$$

where

$$w_{mn}(t,\varepsilon_j) = \frac{e^{-\mu_{mn}^2\varepsilon_j t} - e^{-\mu_{mn}^2\varepsilon_j t_0}}{1-e^{-\mu_{mn}^2\varepsilon_j t_0}}\varphi_{mn} + \frac{1-e^{-\mu_{mn}^2\varepsilon_j t}}{1-e^{-\mu_{mn}^2\varepsilon_j t_0}}\psi_{mn}$$

Hence, for the function $u(t, x)$ we get the representation

$$u(t,x) = \sum_{m,n=1}^{\infty} w_{(2m-1)(2n-1)}(t,\varepsilon_1)X_{(2m-1)(2n-1)}(x) + \sum_{m,n=1}^{\infty} w_{(2m-1)2n}(t,\varepsilon_2)X_{(2m-1)2n}(x)$$

$$+\sum_{m,n=1}^{\infty} w_{2m(2n-1)}(t,\varepsilon_3)X_{2m(2n-1)}(x)+\sum_{m,n=1}^{\infty} w_{2m2n}(t,\varepsilon_4)X_{2m2n}(x) \ . \quad (40)$$

After similar calculations for functions *f*(*x*), we obtain the representation

$$f(x)=\sum_{m,n=1}^{\infty} f_{(2m-1)(2n-1)}(\varepsilon_1)X_{(2m-1)(2n-1)}(x)+\sum_{m,n=1}^{\infty} f_{(2m-1)2n}(\varepsilon_2)X_{(2m-1)2n}(x)$$

$$+\sum_{m,n=1}^{\infty} f_{2m(2n+1)}(\varepsilon_3)X_{2m(2n+1)}(x)+\sum_{m,n=1}^{\infty} f_{2m2n}(\varepsilon_4)X_{2m2n}(x) \ , \quad (41)$$

where

$$f_{mn}(\varepsilon_j)=\mu_{mn}^2\varepsilon_j\left(\frac{1}{1-e^{-\mu_{mn}^2\varepsilon_j t_0}}\psi_{mn}-\frac{e^{-\mu_{mn}^2\varepsilon_j t_0}}{1-e^{-\mu_{mn}^2\varepsilon_j t_0}}\varphi_{mn}\right),\ j=1,2,3,4.$$

Thus, we have proved the following assertion.

Theorem 9. Let $0<\varepsilon_j,\ j=1,2,3,4,$ $g(t)=1$. If functions $\varphi(x)$ and $\psi(x)$ satisfy conditions (14), then the solution to the IP problem exists, is unique, and can be represented in the form (40) and (41).

The following assertion is proved in a similar way.

Theorem 10. Let $0<\varepsilon_j,\ j=1,2,3,4,$ $g(t)\neq 1, g(t)\in C[0,T]$ and $|g(t)|\geq g_0>0$. If functions $\varphi(x)$ and $\psi(x)$ satisfy conditions (14), then the solution to the **IP** problem exists, is unique, and can be represented as series

$$u(t,x)=\sum_{m,n=1}^{\infty} g_{(2m-1)(2n-1)}(\varepsilon_1)X_{(2m-1)(2n-1)}(x)+\sum_{m,n=1}^{\infty} g_{(2m-1)2n}(\varepsilon_2)X_{(2m-1)2n}(x)$$

$$+\sum_{m,n=1}^{\infty} g_{2m(2n-1)}(\varepsilon_3)X_{2m(2n-1)}(x)+\sum_{m,n=1}^{\infty} g_{2m2n}(\varepsilon_4)X_{2m2n}(x)$$

and

$$f(x)=\sum_{m,n=1}^{\infty} f_{(2m-1)(2n-1)}(\varepsilon_1)X_{(2m-1)(2n-1)}(x)+\sum_{m,n=1}^{\infty} f_{(2m-1)2n}(\varepsilon_2)X_{(2m-1)2n}(x)$$

$$+\sum_{m,n=1}^{\infty} f_{2m(2n-1)}(\varepsilon_3)X_{2m(2n-1)}(x)+\sum_{m,n=1}^{\infty} f_{2m2n}(\varepsilon_4)X_{2m2n}(x) \ ,$$

where

$$g_{mn}(\varepsilon_j)=\left(1-\frac{g_{mn}(t)}{g_{mn}(t_0)}\right)e^{-\mu_{mn}^2\varepsilon_j t}\varphi_{mn}+\frac{g_{mn}(t)}{g_{mn}(t_0)}\psi_{mn} \ ,$$

$$f_{mn}(\varepsilon_j)=\frac{1}{g_{mn}(t_0)}\left(\psi_{mn}-\varphi_{mn}e^{-\mu_{mn}^2\varepsilon_j t_0}\right),\ j=1,2,3,4 \ .$$

In this case, the coefficients $g_{mn}(t)$ are determined by the equalities

$$g_{mn}(t) = \int_0^t g(s) e^{-\mu_{mn}^2 \theta_{mn}(t-s)} ds .$$

where

$$\theta_{mn} = \begin{cases} \varepsilon_1, m = 2i-1, n = 2k-1 \\ \varepsilon_2, m = 2i-1, n = 2k \\ \varepsilon_3, m = 2i, n = 2k-1 \\ \varepsilon_4, m = 2i, n = 2k \end{cases} .$$

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