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CORRECTNESS OF THE MIXED PROBLEM FOR DEGENERATE MULTIDIMENSIONAL ELLIPTIC-PARABOLIC EQUATIONS

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The initial-boundary value problem (Dirichlet problem) for general elliptic-parabolic equations of second order was first posed by G. Fichera. Further investigation of this problem was carried out in the monograph by O.A. Oleinik and E.V. Radkevich and the works by V.N. Vragov. In these works, the authors examined mixed problems for degenerate multidimensional elliptic equations. The articles by S.A. Aldashev focused on the correctness (in the sense of uniqueness of solvability) of the Dirichlet problem in a cylindrical domain for multidimensional elliptic-parabolic equations.

A mixed problem for these equations has not been studied. In this paper, the authors demonstrate the uniqueness of solvability and obtain an explicit representation of the classical solution of the mixed problem for degenerate multidimensional elliptic-parabolic equations. The proposed method allows reducing the problem under study to a mixed problem for a degenerate multidimensional elliptic equation examined by S.A. Aldashev.

Keywords: *correctness, mixed problem, degenerate multidimensional equations, spherical functions.*

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ӨЗГЕШЕЛЕНГЕН КӨП ӨЛШЕМДІ ЭЛЛИПТИКО-ПАРАБОЛАЛЫҚ ТЕҢДЕУЛЕР ҮШІН АРАЛАС ЕСЕПТІҢ ДҰРЫСТЫҒЫ

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Екінші ретті жалпы эллиптикало-параболалық теңдеулер үшін бірінші шеткі есепті (Дирихле есебі) қойылымын алғаш рет Г.Фикера жүзеге асырды. Бұл есепті одан әрі зерттеу О.А.Олейник пен Е.В. Радкевичтің монографиясында және В. Н. Враговтың еңбектерінде келтірілген. Авторлардың еңбектері өзгешеленген көп өлшемді эллиптикалық теңдеулерге арналған аралас есептерді зерттеді. С.А. Алдашевтің мақалаларында көпөлшемді эллиптико-параболалық теңдеулер үшін цилиндрлік аймақтағы Дирихле есебінің дұрыстығы (бір мәнді шешімділік мағынасында) зерттелді.

Белгілі болғандай, бұл теңдеулер үшін аралас есеп зерттелмеген. Бұл жұмыс бір мәнді ажыратымдылықты көрсетеді және өзгешеленген көп өлшемді эллиптико-параболалық теңдеулер үшін аралас есептің классикалық шешімінің айқын көрінісі алынады. Ұсынылған әдіс зерттелетін есепті С.А. Алдашев зерттеген өзгешеленген көп өлшемді эллиптикалық теңдеу үшін аралас есептерге дейін азайтуға мүмкіндік береді.

Түйін сөздер: *есептің дұрыстығы, аралас есеп, өзгешеленген көп өлшемді теңдеулер, сфералық функциялар.*

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КОРРЕКТНОСТЬ СМЕШАННОЙ ЗАДАЧИ ДЛЯ ВЫРОЖДАЮЩИХСЯ МНОГОМЕРНЫХ ЭЛЛИПТИКО-ПАРАБОЛИЧЕСКИХ УРАВНЕНИЙ

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Для общих эллиптико-параболических уравнений второго порядка постановку первой краевой задачи (задачи Дирихле) впервые осуществил Г.Фикера. Дальнейшее изучение этой задачи приведено в монографии О. А.Олейника и Е.В.Радкевича, а также в работах В.Н.Врагова. В работах авторов изучались смешанные задачи для вырождающихся многомерных эллиптических уравнений. В статьях С.А.Алдашева для многомерных эллиптико-параболических уравнений исследовалась корректность (в смысле однозначной разрешимости) задачи Дирихле в цилиндрической области.

Насколько известно, смешанная задача для этих уравнений не изучена. В данной работе показана однозначная разрешимость и получено явное представление классического решения смешанноай задачи для вырождающихся многомерных эллиптико-параболических уравнений. Предложенный метод позволяет свести изучаемую задачу к смешанной задаче для вырождающегося многомерного эллиптического уравнения, исследованной С.А. Алдашевым.

Ключевые слова: *корректность, смешанная задача, вырождающиеся многомерные уравнения, сферические функции.*

1. Introduction. A mixed problem for degenerate multidimensional hyperbolic equations in generalized spaces has been studied by F.T. Baranovsky [1] and M.L. Krasnov [2]. S.A. Aldashev [3] and S.G. Mikhlin [4] have proven its correctness for degenerate multidimensional elliptic equations and obtained an explicit form of the classical solution.

To our knowledge, as applied to degenerate multidimensional elliptic-parabolic equations, these issues have not yet been investigated.

The present study demonstrates the unambiguous solvability and obtains an explicit form of the classical solution of a mixed problem for degenerate multidimensional elliptic-parabolic equations.

2. Problem statement and main results. Let $\Omega_{\alpha\beta}$ be a cylindric domain in the Euclidean space E_{m+1} of points $(x_1,..,x_m,t)$, bounded by the cylinder $\Gamma = \{(x,t) : |x| = 1\}$ and the planes $t = \alpha > 0$ and $t = \beta < 0$, where $|x|$ is the length of the vector $(x_1,..,x_m,t)$.

Let us denote by Ω_α and Ω_β parts of the domain $\Omega_{\alpha\beta}$ and by $\Gamma_\alpha, \Gamma_\beta$ parts of the surface Γ lying in the half-spaces $t > 0$ and $t < 0$, with σ_α being the upper and σ_β being the lower bases of the domain $\Omega_{\alpha\beta}$.

Next, let S be the shared part of the boundaries of domains Ω_α and Ω_β that represents the sets $\{t = 0, 0 < |x| < 1\}$ in E_m.

In the domain $\Omega_{\alpha\beta}$, we consider the degenerate multidimensional elliptic-parabolic equations:

$$0 = \begin{cases} p(t)\Delta_x u + u_{tt} + \sum_{i=1}^{m} a_i(x,t)u_{x_i} + b(x,t)u_t + c(x,t)u, t > 0, \\ q(t)\Delta_x u - u_t + \sum_{i=1}^{m} d_i(x,t)u_{x_i} + e(x,t)u, t < 0, \end{cases} , \quad (1)$$

where $p(t) > 0$ for $t > 0, p(0) = 0, p(t) \in C([0,\alpha]) \cap C^2((0,\alpha)), g(t) > 0$ for $t > 0$, $g(0) = 0, g(t) \in C([\beta,0])$, and Δ_x is the Laplace operator of the variables $x_1,\ldots,x_m, m \geq 2$.

Hereinafter, it is convenient to switch from the Cartesian coordinates $x_1,\ldots,x_m,t$ to the spherical ones $r,\theta_1,...,\theta_{m-1},t, r \geq 0, 0 \leq \theta_1 < 2\pi$, $0 \leq \theta_i \leq \pi, i = 2,3,...,m-1, \theta = (\theta_1,...,\theta_{m-1})$.

Problem 1. Find the solution of the equation (1) in the domain $\Omega_{\alpha\beta}$ for $t \neq 0$ from the class $C(\overline{\Omega}_{\alpha\beta}) \cap C^1(\Omega_\alpha) \cap C^2(\Omega_\alpha \cup \Omega_\beta)$ that satisfies the following boundary conditions:

$$u\big|_{\Gamma_\alpha} = \psi_1(t,\theta), \quad (2)$$

$$u\big|_{\Gamma_\beta} = \psi_2(t,\theta), u\big|_{\sigma_\beta} = \varphi(t,\theta), \quad (3)$$

where $\psi_1(0,\theta) = \psi_2(0,\theta), \psi_2(\beta,\theta) = \varphi(1,\theta)$.

Let $\{Y_{n,m}^k(\theta)\}$ be a system of linearly independent spherical functions of order $n, 1 \le k \le k_n$, $(m-2)!n!k_n = (n+m-3)!(2n+m-2)$, and let $W_2^l(S), l = 0,1,...$ be the Sobolev space.

Thus, there is ([5]).

Lemma 1. Let $f(r,\theta) \in W_2^l(S)$. If $l \ge m-1$, then the series:

$$f(r,\theta) = \sum_{n=0}^{\infty} \sum_{k=1}^{k_n} f_n^k(r) Y_{n,m}^k(\theta), \tag{4}$$

and the series obtained by its differentiation of order $p \le l - m + 1$ converge absolutely and uniformly.

Lemma 2. For $f(r,\theta) \in W_2^l(S)$, it is necessary and sufficient for the coefficients of the series (4) to satisfy the inequalities:

$$\left|f_0^1(r)\right| \le c_1, \quad \sum_{n=1}^{\infty} \sum_{k=1}^{k_n} n^{2l} \left|f_n^k(r)\right|^2 \le c, \quad c_1, c_2 = const$$

Let us denote by $\hat{d}_{in}^k(r,t), d_{in}^k(r,t), \tilde{e}_n^k(r,t), \hat{d}_n^k(r,t), \rho_n^k, \overline{\varphi}_n^k(r), \psi_{1n}^k(t), \psi_{2n}^k(t)$ the coefficients of decomposition of the series (4) of the functions $d_i(r,\theta,t)\rho, d_i\frac{x_i}{r}\rho, e(r,\theta,t)\rho$, $d(r,\theta,t)\rho, \rho(\theta), i = 1,...,m, \varphi(r,\theta)$, $\psi_1(t,\theta)$, $\psi_2(t,\theta)$ respectively, where $\rho(\theta) \in C^\infty(H)$ and H is a unit sphere in E_m.

Let $a_i(r,\theta,t), b(r,\theta,t), c(r,\theta,t) \in W_2^l(\Omega_\alpha) \subset C(\overline{\Omega}_\alpha)$, $d_i(r,\theta,t), e(r,\theta,t) \in W_2^l(\Omega_\beta), i = 1,...,$ $m, l \ge m+1$, $e(r,\theta,t) \le 0, \forall (r,\theta,t) \in \Omega_\beta$.

Then it is correct.

Theorem 1. If $\varphi(r,\theta) \in W_2^\rho(S), \psi_1(t,\theta) \in W_2^\rho(\Gamma_\alpha), \psi_2(t,\theta) \in W_2^l(\Gamma_\beta), \rho > \frac{3m}{2}$, Problem 1 is uniquely solvable.

3. Solvability of Problem 1. First, let us demonstrate the solvability of problem (1), (3). In the spherical coordinates, the equation (1) in the domain Ω_β takes the form:

$$L_1 u \equiv g(t)\left(u_{rr} + \frac{m-1}{r}u_r - \frac{1}{r^2}\delta u\right) - u_t + \sum_{i=1}^{m} d_i(r,\theta,t) u_{x_i} + e(r,\theta,t)u = 0, \tag{5}$$

$$\delta \equiv -\sum_{j=1}^{m-1} \frac{1}{g_j \sin^{m-j-1}\theta_j} \frac{\partial}{\partial\theta_j}\left(\sin^{m-j-1}\frac{\partial}{\partial\theta_j}\right), \quad g_1 = 1, g_i = (\sin\theta_1 ... \sin\theta_{j-1})^2, j > 1$$

It is known [5] that the spectrum of the operator δ consists of eigenvalues $\lambda_n = n(n+ +m-2), n = 0,1,...,$ each of them corresponding to k_n orthonormalized eigenfunctions $Y_{n,m}^k(0)$.

Thus, we can search for the solution of Problem 1 in the domain Ω_β in the form:

$$u(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{k=1}^{k_n} \bar{u}_n^k(r,t) Y_{n,m}^k(\theta), \tag{6}$$

where $\bar{u}_n^k(r,t)$ are the functions to be determined.

Substituting (6) into (5) and multiplying the obtained expression by $\rho(\theta) \neq 0$ and then integrating over the unit sphere H for $\bar{u}_n^k$, we obtain [3, 4]:

$$g(t)\rho_0^1 \bar{u}_{0rr}^1 - \rho_0^1 \bar{u}_{0t}^1 + \left(\frac{m-1}{r} g(t)\rho_0^1 + \sum_{i=1}^{m} d_{i0}^1\right)\bar{u}_{0r}^1 + \tilde{e}_0^1 \bar{u}_0^1 +$$

$$+\sum_{n=1}^{\infty}\sum_{k=1}^{k_n}\left\{ g(t)\rho_n^k \bar{u}_{nrr}^k - \rho_n^k \bar{u}_{nt}^k + \left(\frac{m-1}{r} g(t)\rho_n^k + \sum_{i=1}^{m} d_{nr}^k\right)\bar{u}_{nr}^k + \right. \tag{7}$$

$$\left. + \left[\tilde{e}_n^k - \lambda_n \frac{\rho_n^k}{r^2} g(t) + \sum_{i=1}^{m}(\tilde{d}_{in-1}^k - n d_{in}^k)\right]\bar{u}_n^k \right\} = 0,$$

Next, let us analyze the infinite system of differential equations:

$$g(t)\rho_0^1 \bar{u}_{0rr}^1 - \rho_0^1 \bar{u}_{0t}^1 + \frac{(m-1)}{r} g(t)\rho_0^1 \bar{u}_{0r}^1 = 0, \tag{8}$$

$$g(t)\rho_1^k \bar{u}_{1rr}^k - \rho_1^k \bar{u}_{1t}^k + \frac{(m-1)}{r} g(t)\rho_1^k \bar{u}_{1r}^k - \frac{\lambda_1}{r^2} g(t)\rho_1^k \bar{u}_1^k = -\frac{1}{k_1}\left(\sum_{i=1}^{m} d_{i0}^1 \bar{u}_{0r}^1 + \tilde{e}_0^1 \bar{u}_0^1\right), n=1, k=\overline{1,k_1},$$

$$g(t)\rho_n^k \bar{u}_{nrr}^k - \rho_n^k \bar{u}_{nt}^k + \frac{(m-1)}{r} g(t)\rho_n^k \bar{u}_{nr}^k - \frac{\lambda_n}{r^2} g(t)\rho_n^k \bar{u}_n^k = -\frac{1}{k_n}\sum_{k=1}^{k_{n-1}}\left\{\sum_{i=1}^{m} d_{in-1}^k \bar{u}_{n-1r}^k + \right. \tag{9}$$

$$\left. + \left[\tilde{e}_{n-1}^k + \sum_{i=1}^{m}(\tilde{d}_{in-2}^k - (n-1)d_{in-1}^k\right]\bar{u}_{n-1}^k\right\}, k=\overline{1,k_n}, n=2,3,...,$$

If $\{\bar{u}_n^k\}, k=\overline{1,k_n}, n=0,1,...$ is the solution of the system (8), (9), it is also the solution of the equation (7).

It is easy to notice that each equation in the system (8), (9) can be presented in the form:

$$g(t)\left(\bar{u}_{nrr}^k + \frac{(m-1)}{r}\bar{u}_{nr}^k - \frac{\lambda_n}{r^2}\bar{u}_n^k\right) - \bar{u}_{nt}^k = \bar{f}_n^k(r,t), \tag{10}$$

where $\overline{f}_n^k(r,t)$ are determined from the previous equations of this system with $\overline{f}_n^k(r,t) \equiv 0$.

Next, from the boundary conditions (3) and in view of (6) we obtain:

$$\overline{u}_n^k(r,\beta) = \overline{\varphi}_n^k(r), \overline{u}_n^k(1,t) = \psi_{2n}^k(t), k = \overline{1,k_n}, n = 0,1,\ldots, \quad (11)$$

In (10), (11), making a change of variables $\overline{\upsilon}_n^k(r,t) = \overline{u}_n^k(r,t) - \psi_n^k(t)$, we obtain:

$$g(t)\left(\overline{\upsilon}_{nrr}^k + \frac{(m-1)}{r}\overline{\upsilon}_{nr}^k - \frac{\lambda_n}{r^2}\overline{\upsilon}_n^k\right) - \overline{\upsilon}_{nt}^k = f_n^k(r,t), \quad (12)$$

$$\overline{\upsilon}_n^k(r,\beta) = \varphi_n^k(r), \overline{\upsilon}_n^k(1,t) = 0, k = \overline{1,k_n}, n = 0,1,\ldots, \quad (13)$$

$$f_n^k(r,t) = \overline{f}_n^k(r,t) + \psi_{2nt}^k + \frac{\lambda_n g(t)}{r^2}\psi_{2nt}^k,\ \varphi_n^k(r) = \overline{\varphi}_n^k(r) - \psi_{2n}^k(\beta)$$

Making a change of variable $\overline{\upsilon}_n^k(r,t) = r^{\frac{(1-m)}{2}}\upsilon_n^k(r,t)$, problem (12), (13) can be reduced to the following problem:

$$L\upsilon_n^k \equiv g(t)\left(\upsilon_{nrr}^k + \frac{\overline{\lambda}_n}{r^2}\upsilon_n^k\right) - \upsilon_{nt}^k = \tilde{f}_n^k(r,t), \quad (14)$$

$$\upsilon_n^k(r,\beta) = \tilde{\varphi}_n^k(r), \upsilon_n^k(1,t) = 0, \quad (15)$$

$$\overline{\lambda}_n = \frac{(m-1)(3-m) - 4\lambda_n}{4}, \tilde{f}_n^k(r,t) = r^{\frac{(m-1)}{2}} f_n^k(r,t), \tilde{\varphi}_n^k(r) = r^{\frac{(m-1)}{2}}\varphi_n^k(r)$$

The solution of the problem (14), (15) is sought in the form:

$$\upsilon_n^k(r,t) = \upsilon_{1n}^k(r,t) + \upsilon_{2n}^k(r,t), \quad (16)$$

where $\upsilon_{1n}^k(r,t)$ is the solution to the problem:

$$L\upsilon_{1n}^k = \tilde{f}_n^k(r,t), \quad (17)$$

$$\upsilon_{1n}^k(r,\beta) = 0, \upsilon_{1n}^k(1,t) = 0, \quad (18)$$

and $\upsilon_{2n}^k(r,t)$ is the solution to:

$$L\upsilon_{2n}^k = 0, \quad (19)$$

$$\upsilon_{2n}^k(r,\beta) = \tilde{\varphi}_n^k(r),\ \upsilon_{2n}^k(1,t) = 0, \quad (20)$$

The solutions to the above problems are analyzed in the form:

$$\upsilon_n^k(r,t) = \sum_{s=1}^{\infty} R_s(r)T_s(t)\,, \tag{21}$$

in this, let:

$$\tilde{f}_n^k(r,t) = \sum_{s=1}^{\infty} a_{s,n}(t)R_s(r), \tilde{\varphi}_n^k(r) = \sum_{s=1}^{\infty} b_{s,n}(t)R_s(r), \tag{22}$$

Substituting (21) into (17), (18), and considering (22), we reach the problem:

$$R_{srr} + \frac{\overline{\lambda_n}}{r^2}R_s + \mu R_s = 0, 0 < r < 1\,, \tag{23}$$

$$R_s(1) = 0, \left|R_s(0)\right| < \infty\,, \tag{24}$$

$$T_{st} + \mu g(t)T_s = -a_{s,n}(t), \beta < t < 0, \tag{25}$$

$$T_s(\beta) = 0\,, \tag{26}$$

The bounded solution of the problem (23), (24) is [6]:

$$R_s(r) = \sqrt{r}J_\upsilon(\mu_{s,n}r)\,, \tag{27}$$

where $\upsilon = n + \frac{(m-2)}{2}, \mu = \mu_{s,n}^2$.

The solution to the problem (25), (26) is:

$$T_{s,n}(t) = (\exp(-\mu_{s,n}^2 \int_0^t g(\xi)d\xi))(\int_t^\beta a_{s,n}(\xi)(\exp \mu_{s,n}^2 \int_0^\xi g(\xi_1)d\xi_1)d\xi)\,, \tag{28}$$

Substituting (27) into (22), we obtain:

$$r^{-\frac{1}{2}}\tilde{f}_n^k(r,t) = \sum_{s=1}^{\infty} a_{s,n}(t)J_\upsilon(\mu_{s,n}r), r^{-\frac{1}{2}}\tilde{\varphi}_n^k(r,t) = \sum_{s=1}^{\infty} b_{s,n}J_\upsilon(\mu_{s,n}r), 0 < r < 1, \tag{29}$$

The series (29) are the decompositions into the Fourier-Bessel series [7] if:

$$a_{s,n}(t) = 2[J_{\upsilon+1}(\mu_{s,n})]^{-2}\int_0^1 \sqrt{\xi}\tilde{f}_n^k(\xi,t)J_\upsilon(\mu_{s,n}\xi)d\xi\,, \tag{30}$$

$$b_{s,n} = 2\left[J_{\upsilon+1}(\mu_{s,n})\right]^{-2}\int_0^1 \sqrt{\xi}\tilde{\varphi}_n^k(\xi)J_\upsilon(\mu_{s,n}\xi)d\xi\,, \tag{31}$$

where $\mu_{s,n}, s = 1,2,...,$ are the positive zeros of the Bessel functions $J_\upsilon(z)$ put in the increasing order.

From (21), (27), (28) we obtain the solution to the problem (17), (18):

$$\upsilon_{1n}^k(r,t)=\sum_{s=1}^{\infty}\sqrt{r}T_{s,n}(t)J_\upsilon(\mu_{s,n}r)\,, \tag{32}$$

where $a_{s,n}(t)$ is determined from (30).

Next, substituting (21) into (19), (20) and considering (22), we obtain the problem:

$$T_{st}+\mu_{s,n}^2 g(t)T_s=0, \beta<t<0, T_s(\beta)=b_{s,n}$$

the solution to which is:

$$T_{s,n}(t)=b_{s,n}\exp\left(\mu_{s,n}^2\int_t^\beta g(\xi)d\xi\right), \tag{33}$$

From (27), (33) we obtain:

$$\upsilon_{2n}^k(r,t)=\sum_{s=1}^{\infty}b_{s,n}\sqrt{r}\left(\exp\mu_{s,n}^2\int_t^\beta g(\xi)d\xi\right)J_\upsilon(\mu_{s,n}r)\,, \tag{34}$$

where $b_{s,n}$ is determined from (31).

Thus, by first solving the problem (8), (11) (n = 0), and then (9), (11), (n = 1), etc., we are able to consecutively find all functions $\upsilon_n^k(r,t)$ from (16), where $\upsilon_{1n}^k(r,t)$, $\upsilon_{2n}^k(r,t)$ are defined from (32) and (34).

Thus, in the domain Ω_β, there is:

$$\int_H \rho(\theta)L_1 u dH=0\,, \tag{35}$$

Let $f(r,\theta,t)=R(r)\rho(\theta)T(t)$, with $R(r)\in V_0$, being dense in $L_2((0,1)),\rho(\theta)\in C^\infty(H)$ being dense in $L_2(H)$, and $T(t)\in V_1$, V_1 – in $L_2((\beta,0))$. Then $f(r,\theta,t)\in V$, $V=V_0\otimes H\otimes V_1$ is dense in $L_2(\Omega_\beta)$ [8].

From this and (35), it follows that:

$$\int_{\Omega_\beta} f(r,\theta,t)L_1 u d\Omega_\beta=0$$

and

$$L_1 u=0\,,\ \forall(r,\theta,t)\in\Omega_\beta$$

Thus, the solution to the problem (1), (3) in the domain Ω_β is the function:

$$u(r,\theta,t)=\sum_{n=0}^{\infty}\sum_{k=1}^{k_n}\left\{\psi_{2n}^k(t)+r^{\frac{(1-m)}{2}}\left[\upsilon_{1n}^k(r,t)+\upsilon_{2n}^k(r,t)\right]\right\}Y_{n,m}^k(\theta)\,, \tag{36}$$

where $\upsilon_{1n}^k(r,t),\upsilon_{2n}^k(r,t)$ is determined from (32), (34).

Considering the formula [7] $2J_\upsilon'(z)=J_{\upsilon-1}(z)-J_{\upsilon+1}(z)$, the estimates [5, 9]:

$$J_\upsilon(z)=\sqrt{\frac{2}{\pi z}}\cos\left(z-\frac{\pi}{2}\upsilon-\frac{\pi}{4}\right)+0\left(\frac{1}{z^{3/2}}\right),\upsilon\geq 0$$

$$k_n\leq c_1 n^{m-2},\left|\frac{\partial^q}{\partial\theta_j^q}Y_{n,m}^k(\theta)\right|\leq c_2 n^{\frac{m}{2}-1+l},j=\overline{1,m-1},l=0,1,..., \tag{37}$$

as well as Lemmas and the bounds on the set functions $\psi_2(t,\theta),\varphi(t,\theta)$, same as in [10], we can demonstrate that the obtained solution (36) belongs to the class $C(\overline{\Omega}_\beta)\cap C^2(\Omega_\beta)$.

Next, from (32), (34), (36) and with $t\to -0$, we obtain:

$$u(r,\theta,0)=\tau(r,\theta)=\sum_{n=0}^{\infty}\sum_{k=1}^{k_n}\tau_n^k(r)Y_{n,m}^k(\theta), \tag{38}$$

$$\tau_n^k(r)=\psi_{2n}^k(0)+\sum_{s=1}^{\infty}r^{\frac{(2-m)}{2}}\left[\int_0^\beta a_{s,n}(\xi)(\exp\mu_{s,n}^2\int_0^\xi g(\xi_1)d\xi_1)d\xi+\right.$$

$$\left.+b_{s,n}\exp(\mu_{s,n}^2\int_0^\beta g(\xi)d\xi)\right]J_{n+\frac{(m+2)}{2}}(\mu_{s,n}r)$$

$$u_t(r,\theta,0)=\upsilon(r,\theta)=\sum_{n=0}^{\infty}\sum_{k=1}^{k_n}\upsilon_n^k(r)Y_{n,m}^k(\theta), \tag{39}$$

$$\upsilon_n^k(r)=\psi_{2nt}^k(0)-\sum_{s=1}^{\infty}r^{\frac{(2-m)}{2}}\left[a_{s,n}(0)+\mu_{s,n}^2 g(0)\int_0^\alpha a_{s,n}(\xi)\left(\exp(\mu_{s,n}^2\int_0^\xi g(\xi_1)d\xi_1)\right)d\xi+\right.$$

$$\left.+\mu_{s,n}^2 g(0)b_{s,n}(\exp\mu_{s,n}^2\int_0^\alpha g(\xi)d\xi)\right]J_{n+\frac{(m+2)}{2}}(\mu_{s,n}r)$$

From (30)-(32), (34), as well as the lemmas, it follows that $\tau(r,\theta),\upsilon(r,\theta)\in W_2^l(S),l>\frac{3m}{2}$.

Thus, considering the boundary conditions (2), (38), (39) in the domain Ω_α, we come to the mixed problem for degenerate elliptic equations:

$$L_2 u\equiv p(t)\Delta_x u+u_{tt}+\sum_{i=1}^{m}a_i(r,\theta,t)u_{x_i}+b(r,\theta,t)u_t+c(r,\theta,t)u=0, \tag{40}$$

with the conditions:

$$u\big|_S=\tau(t,\theta),u_t\big|_S=\upsilon(r,\theta),u\big|_{\Gamma_\alpha}=\psi_1(t,\theta), \tag{41}$$

The following theorem is proven in [4].

Theorem 2. If $\tau(r,\theta),\upsilon(r,\theta)\in W_2^l(S)$, $\psi_1(t,\theta)\in W_2^l(\Gamma_\alpha)$, $l>\frac{3m}{2}$, then the problem (40), (41) has a unique solution.

Next, using Theorem 2, we come to the solvability of Problem 1.

Uniqueness of the solution of Problem 1. First, we shall consider the problem (1), (3) in the domain Ω_β and prove the uniqueness of its solution. For this, we construct the solution of the first boundary value problem for the equation:

$$L_1^*\upsilon\equiv g(t)\Delta_x\upsilon-\upsilon_t-\sum_{i=1}^m d_i\upsilon_{x_i}+d\upsilon=0\,, \tag{5*}$$

with the conditions:

$$\upsilon\big|_S=\tau(r,\theta)=\bar{\tau}_n^k(r)Y_{n,m}^k(\theta),\ \upsilon\big|_{\Gamma_\beta}=0\,, \tag{42}$$

$d(x,t)=e-\sum_{i=1}^m d_{ix_i}$, where $\bar{\tau}_n^k(r)\in W$, W is the set of functions $\tau(r)$ from the class $C([0,1])\cap C^1((0,1))$. The set W is dense everywhere in $L_2((0,1))$ [8]. The solution to problem (5*), (42) will be found in the form (6), where the functions $\bar{\upsilon}_n^k(r,t)$ will be defined below. Then, by analogy to the previous section, the functions $\bar{\upsilon}_n^k(r,t)$ satisfy the system of equations (8), (9), where $\tilde{d}_{in}^k, d_{in}^k$ are replaced with $-\tilde{d}_{in}^k$, $-d_{in}^k$, and $\tilde{e}_n^k$ is replaced with $\tilde{d}_n^k, i=1,\dots,m, k=\overline{1,k_n}, n=0,1,\dots$.

Next, from the boundary condition (42) and given (6), we come to the following problem:

$$L_1\upsilon_n^k\equiv g(t)\left(\upsilon_{nrr}^k+\frac{\bar{\lambda}_n}{r^2}\upsilon_n^k\right)-\upsilon_{nt}^k=\tilde{f}_n^k(r,t)\,, \tag{43}$$

$$\upsilon_n^k(r,0)=\tau_n^k(r),\ \upsilon_n^k(1,t)=0\,, \tag{44}$$

$$\upsilon_n^k(r,t)=r^{\frac{(m-1)}{2}}\bar{\upsilon}_n^k(r,t),\ \tilde{f}_n^k(r,t)=r^{\frac{(m-1)}{2}}\bar{f}_n^k(r,t),\ \tau_n^k(r)=r^{\frac{(m-1)}{2}}\bar{\tau}_n^k(r)$$

Thus, we built the solution to the problem (5*), (42) in the form of the series:

$$\upsilon(r,\theta,t)=\sum_{n=0}^\infty\sum_{k=1}^{k_n}r^{\frac{(1-m)}{2}}[\upsilon_{1n}^k(r,t)+\upsilon_{2n}^k(r,t)]Y_{n,m}^k(\theta)$$

which, given the estimates (37), belongs to the class $C(\bar{\Omega}_\beta)\cap C^1(\Omega_\beta\cup S)\cap C^2(\Omega_\beta)$.

By integrating by the domain Ω_β the equation [11]:

$$\upsilon L_1u-uL_1^*\upsilon=-\upsilon P(u)+uP(\upsilon)-u\upsilon Q$$

where:

$$P(u) = g(t)\sum_{i=1}^{m} u_{x_i} \cos(N^{\perp}, x_i), Q = \cos(N^{\perp}, t) - \sum_{i=1}^{m} a_i \cos(N^{\perp}, x_i)$$

and $N^{\perp}$ is the inner normal to the boundary $\partial\Omega_{\beta}$, using the Green's formula, we obtain:

$$\int_S \tau(r,\theta)u(r,\theta,0)ds = 0\,, \tag{45}$$

Given that the linear hull of the system of the functions $\left\{\bar{\tau}_n^k(r)Y_{n,m}^k(\theta)\right\}$ is dense in $L_2(S)$ [8], we can conclude from (45) that $u(r,\theta,0) = 0, \forall(r,\theta) \in S$.

Hence, by the extremum principle for the parabolic equation (5) [12] $u \equiv 0$ in $\overline{\Omega}_{\beta}$.

It follows from this that $u_t(r,\theta,0) = \upsilon(r,\theta) = 0, \forall(r,\theta) \in S$.

Thus, we arrived at the homogeneous mixed problem (40), (41), which by Theorem 2 has a trivial solution.

Therefore, the uniqueness of the solution to Problem 1 is established.

Since in [4] we obtained an explicit form of the solution of problem (40), (41), we can also write an explicit representation for Problem 1.

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