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ON THE QUADRATIC CLOSENESS OF EIGENFUNCTIONS OF «UNPERTURBED» AND «PERTURBED» DIFFERENTIATION OPERATORS ON AN INTERVAL

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The article considers the eigenvalue problem of the differentiation operator, when the spectral parameter is also present in the boundary condition with an integral perturbation, where the integrand has the property of limited variation and has a value of unity at the ends of the segment [-1, 1]. The derivatives with respect to the time variable included in the boundary condition naturally arise when solving (by the Fourier method) initial boundary value problems for evolution equations. The conjugate operator is constructed. It is shown that the spectral questions of the conjugate operator have a similar structure. The characteristic determinant of the original direct spectral problem with an integral perturbation of the boundary condition and for the eigenvalue problem of a loaded first-order differential equation on a segment with a periodic boundary condition, which is an entire analytical function of the spectral parameter, is constructed. Based on the formula of the characteristic determinant, conclusions are drawn about the asymptotic behavior of the spectrum of the original «perturbed» differentiation operator and the loaded first-order differential equation on the segment. A special feature of the operator under consideration is the non-self-adjointness of the operator in $L_2(-1,1)$. The quadratic proximity of the systems of eigen functions of the «unperturbed» and «perturbed» differentiation operators and the Riesz basis property of these systems are proved. In this case, the system of eigenfunctions of the «perturbed» operator is not orthonormal.

Keywords: *quadratic closeness, Riesz basis, orthonormality, operator, perturbation, loaded.*

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КЕСІНДІДЕГІ «ТОЛҚЫТЫЛМАҒАН» ЖӘНЕ «ТОЛҚЫТЫЛҒАН» ДИФФЕРЕНЦИАЛДАУ ОПЕРАТОРЛАРЫНЫҢ МЕНШІКТІ ФУНКЦИЯЛАР ЖҮЙЕЛЕРІНІҢ КВАДРАТТЫҚ ЖАҚЫНДЫҒЫ ТУРАЛЫ

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Бұл мақалада дифференциалдау амалынан туындайтын оператордың шеттік шарттары спектралдық параметрмен беріліп, интегралдық толқытылғандағы спектралдық есебі қарастырылған. Интеграл астындағы функция өзгеруі шенелген болғандығы, [*–1,1*] *кесіндісінің шеткі нүктелеріндегі мәні бір болғандағы «толқытылмаған» және «толқытылған» меншікті функциялар жүйелерінің квадраттық жақындығы дәлелденіп, Рисс базистілігі көрсетілген және ортонормаланған жүйе болмайтындығы көрсетілген. Шеттік жағдайдағы уақыт айнымалысына қатысты туындылар эволюция теңдеулері үшін бастапқы шекаралық есептерді шешу кезінде (Фурье әдісі бойынша) пайда болады. Түйіндес оператор құрылған. Шеттік шартта интегралдық «толқытылуы» бар бастапқы тікелей спектралдық есептің және периодтық шеттік шартпен берілген бірінші ретті жүктелген дифференциалдық оператордың характеристикалық анықтауышы құрылған. Характеристикалық анықтауыштың формуласының негізінде кесіндідегі бастапқы тікелей «толқытылған» оператордың және жүктелген түйіндес оператордың меншікті мәндерінің асимптотикасы туралы қорытынды жасалады.Қарастырылып отырған есептің ерекшелігі, оның* $L_2(-1,1)$ *кеңістігіндегі өзіне-өзі түйіндес болмайтындығында болып табылады.*

Түйін сөздер: *квадраттық жақындық, Рисс базисі, ортонормаланғандық, оператор, толқыту, жүктелгендік.*

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О КВАДРАТИЧНОЙ БЛИЗОСТИ СОБСТВЕННЫХ ФУНКЦИЙ «НЕВОЗМУЩЕННОГО» И «ВОЗМУЩЕННОГО» ОПЕРАТОРОВ ДИФФЕРЕНЦИРОВАНИЯ НА ОТРЕЗКЕ

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В настоящей статье рассматривается задача на собственные значения оператора дифференцирования, когда спектральный параметр также присутствует и в краевом условии с интегральным возмущением, где подынтегральная функция обладает свойством ограниченной вариации и на концах отрезка [–1,1] имеет значение единица. В краевое условие входящие производные по временной переменной естественным образом возникают при решении (методом Фурье) начально-краевых задач для эволюционных уравнений. Построен сопряженный оператор. Показано, что спектральные вопросы сопряженного оператора имеют аналогичную структуру. Построен характеристический определитель исходной прямой спектральной задачи с интегральным возмущением краевого условия и для задачи на собственное значение нагруженного дифференциального уравнения первого порядка на отрезке с периодическим краевым условием, которое является целой аналитической функцией от спектрального параметра. На основе формулы характеристического определителя делаются выводы об асимптотике спектра исходного «возмущенного» оператора дифференцирования и нагруженного дифференциального уравнения первого порядка на отрезке. Особенностью рассматриваемого оператора является несамосопряженность оператора в $L_2(-1,1)$.

Доказывается квадратичная близость систем собственных функций «невозмущённой» и «возмущённой» операторов дифференцирования и базисность Рисса этих систем. При этом система собственных функций «возмущенного» оператора не является ортонормированным.

Ключевые слова: *квадратичная близость, базис Рисса, ортонормированность, оператор, возмущение, нагруженный.*

Introduction. The system of eigenfunctions of an operator, formally defined by a self-adjoint differential expression with arbitrary self-adjoint boundary conditions ensuring a discrete spectrum, forms an orthogonal basis in the space L_2. In cases where the spectral parameter is also present in the boundary condition, considering the operator problem in L_2 becomes impossible, as the spectral parameter should not alter the domain of the operator.

Problems of this type naturally arise when solving initial-boundary value problems for evolutionary equations using the Fourier method, especially when the boundary condition involves derivatives of solutions with respect to the time variable. For such problems, based on the theory developed by A.A. Shkalikov [1], it is known that by removing a finite number of elements, the system of eigenfunctions and the system of adjoint functions of such a problem can form a basis with brackets. One of the latest developments in this area is mentioned in [2].

As demonstrated in [3], the system of eigenfunctions and the system of adjoint functions of the Samarskii-Ionkin type problem forms a Riesz basis in $L_2(0,1)$.

Formulation of the problem and the main result. In the work of V.A. Ilyin [4], it was first noted that a small change in the values of the coefficients of the equations will change the basic properties of the root functions. This idea was developed in the work of A.S. Makin [5] for the case of a non-self-adjoint perturbation of a self-adjoint periodic problem, and for a loaded second-order differential operator with periodic boundary conditions in the works [6], [7], [8], [9]. In the work of I.S. Lomov [10], it was possible to extend the method of spectral expansions by V.A. Ilyin [4] to the case of loaded differential operators.

In [11], the asymptotics of the eigenvalues and corresponding eigenfunctions of the first-order differential operator are found in the case when the spectral parameter is also present in the boundary condition with an integral perturbation in $L_2(-1,1)$, i.e.

$$L_1 u = u'(t) = \lambda u(t), \quad -1 < t < 1, \tag{2.1}$$

$$u(-1) = u(1) + \lambda \int_{-1}^{1} u(t)\Phi(t)\,dt, \tag{2.2}$$

where $\Phi(t)$ is a function of bounded variation and satisfies the condition

$$\Phi(-1) = \Phi(1) = 1, \tag{2.3}$$

λ is a complex number representing the spectral parameter, turning the problem (2.1)-(2.3) into a non-self-adjoint problem. Another variant is perturbation, the conjugate problem is a loaded first-order differential operator in $L_2(-1,1)$. In [12], [13] it is proved that the asymptotics of the eigenvalues and eigenfunctions of the conjugate problem has a similar structure. For clarity, we reproduce the main results of the works [11], [12].

Theorem 2.1. [11]. If $\Phi(t)$ is a function of bounded variation and $\Phi(-1) = \Phi(1) = 1$, then all eigenvalues of the «perturbed» differentiation operator belong to the strip $|\mathrm{Re}\,\lambda| = |x| < k$, for some k, where $\lambda = x + iy$.

Theorem 2.2. [12]. Let $\Phi(t)$ be a function of bounded variation and satisfy condition (2.3). Then the eigenvalues of the «perturbed» operator L_1 form a countable set and the asymptotics has the form $\lambda_n^1 = i\pi n + \underset{=}{O}(1)$ as $n \to \infty$ and the corresponding eigenfunctions of the operators L_1 and L_1^* can be represented as

$$u_n^{(1)}(t) = v_n^{(1)}(t) \approx C \cdot e^{in\pi t} \cdot e^{\varepsilon t}, \quad \forall C > 0.$$

The problem statement and the main result of this work involve the investigation of the basis properties of the systems of eigenfunctions $\left\{u_n^{(1)}(t)\right\}$ and $\left\{v_n^{(1)}(t)\right\}$ of operators L_1 and L_1^*. These systems exhibit fundamental differences from previous works [11], [12], [13], following the generalization of the spectral nature.

In the case when $\Phi(t) \equiv 0$, an «undisturbed» spectral problem arises as

$$L_0 u = u'(t) = \lambda u(t), \quad -1 < t < 1, \quad u(-1) = u(1) \tag{2.4}$$

The characteristic determinant of the spectral problem (2.4) is given by $\Delta_0(\lambda) = e^{-\lambda} - e^{\lambda}$, an entire function in the form of an exponential quasipolynomial. Entire functions of this class have been studied in many classical works [14], [15], [16].

The zeros of the integer function $\Delta_0(\lambda)$ are the numbers $\lambda_n^0 = in\pi$, $n = 1,2,3,...$, which adequately define the eigenvalues of the operator L_0. In this case, the eigenfunctions of the «undisturbed» operator are $u_{n0}^0(t) = C \cdot e^{in\pi t}$, $\forall C > 0$, which forms a complete orthonormal system [17], [18] and form a Riesz basis in $L_2(-1,1)$ [19].

It will be demonstrated that the function systems and are quadratically close in $L_2(-1,1)$.

Indeed, we have [17]

$$\sum_{n=1}^{\infty} \left\| u_n^1 - u_{n0}^0 \right\|^2 = \sum_{n=1}^{\infty} \left\| v_n^1 - u_{n0}^0 \right\|^2 .$$

By decomposition $e^{\lambda} - 1 = \lambda + \frac{\lambda^2}{2!} + ...$, therefore $e^{\lambda} - 1 = \underset{=}{O}(\lambda)$. Direct computation yields the estimate $\left\| u_n^1 - u_{n0}^0 \right\| = \left\| e^{i\pi n t} \cdot \left(e^{\varepsilon t} - 1 \right) \right\| \leq C \cdot \left\| \varepsilon t \right\| = \underset{=}{O}\left(\frac{1}{n} \right)$.

Taking into account the asymptotics of $\underset{=}{O}\left(\frac{1}{n} \right)$, we obtain that

$$\sum_{n=1}^{\infty} \left\| u_n^1 - u_{n0}^0 \right\|^2 = \sum_{n=1}^{\infty} \left\| v_n^1 - u_{n0}^0 \right\|^2 \leq C \cdot \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

that is, systems of functions $\left\{ u_n^1(t) \right\}, \left\{ v_n^1(t) \right\}$ and $\left\{ u_{n0}^0(t) \right\}$ are quadratically close.

Thus proved

Theorem 2.3. The systems of eigenfunctions $\left\{ u_n^1(t) \right\}, \left\{ v_n^1(t) \right\}$, of the «perturbed» operators L_1 and L_1^* are quadratically close to the system of eigenfunctions $\left\{ u_{n0}^0(t) \right\}$ of the «unperturbed» operator L_0.

From this, it follows that

Corollary 2.1. The system $\left\{ u_n^1(t) \right\}$, consequently, the system $\left\{ v_n^1(t) \right\}$ form a Riesz basis in $L_2(-1,1)$, but they are not orthonormal.

Spectral issues of a first-order differential operator in terms of the completeness of the system of eigenfunctions was studied in the monograph by A. Shaldanbayev [20]. For a third-order operator, issues related to the study of zeros of the entire functions in the form of quasipolynomials were explored in the works [21], [22], [23]. The question of the basisness of the Sturm-Liouville operator with integral perturbation of one of the boundary conditions was investigated in the works [24], [25], [26].

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